Estimates for Boundary Blowup Solutions of $p$-Laplacian Type Quasilinear Elliptic Equations

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Abstract

In this paper, we investigate the effect of the mean curvature of the boundary $\partial \Omega$ on the behavior of the blow-up solutions to the $p$-Laplacian type quasilinear elliptic equation

$$\text{div}(|\nabla u|^{p-2}\nabla u) = u^m|\nabla u|, \quad p > 1,$$

where $\Omega \subseteq \mathbb{R}^N$ be a bounded smooth domain. Under appropriate conditions on $p$ and $m$, we find the estimates of the solution $u$ in terms of the distance from $x$ to the boundary $\partial \Omega$. To the equation

$$\text{div}(|\nabla u|^{p-2}\nabla u) = u^m|\nabla u|^q, \quad p > 1, \quad 0 < q < 1,$$

the results of the semilinear problem are extended to the quasilinear ones.

Keywords: $p$-Laplacian elliptic equation; boundary blow-up solution; estimates.

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1 Introduction

In this paper, we study the boundary blow-up problems

\[ \text{div}(|\nabla u|^{p-2} \nabla u) = u^m |\nabla u| \quad \text{in} \quad \Omega, \quad u \to \infty \quad \text{as} \quad x \to \partial \Omega, \]  

(1.1)

and

\[ \text{div}(|\nabla u|^{p-2} \nabla u) = u^m |\nabla u|^q \quad \text{in} \quad \Omega, \quad u \to \infty \quad \text{as} \quad x \to \partial \Omega, \]  

(1.2)

where \( \Omega \) is a bounded smooth domain in \( \mathbb{R}^N \), \( N \geq 2 \), \( p > 1 \), \( m + 1 > p - 1 \), and \( 0 < q < 1 \).

First we consider to prove the existence of a positive large solution. We first consider, for \( 0 < \varepsilon < 1 \), the problem

\[ \Delta u = u^m \quad \text{in} \quad \Omega, \quad u \to \infty \quad \text{as} \quad x \to \infty, \]  

(1.3)

C. Bandle in [8] has found the estimate

\[ u(x) = \left( \frac{p - 1}{\sqrt{2(p + 1)}} \delta(x)^{\frac{2}{p-1}} \right)^{\frac{2}{p}} = \left[ 1 + \frac{(N - 1)H(\pi)}{p + 3} + o(\delta(x)) \right], \]  

(1.4)

where \( \delta(x) \) denotes the distance from \( x \) to the boundary \( \partial \Omega \), and \( H(\pi) \) denotes the mean curvature of \( \partial \Omega \) at the point \( \pi \) nearest to \( x \).

In [11], the authors investigate the problem

\[ \Delta u = u^p |\nabla u|^q \quad \text{in} \quad \Omega, \quad u \to \infty \quad \text{as} \quad x \to \infty. \]  

(1.5)

where \( \Omega \) is a bounded smooth domain in \( \mathbb{R}^N \), \( N \geq 2 \), \( p > 0 \), \( 0 \leq q \leq (p + 3)/(p + 2) \) and \( p + q > 1 \). They find an estimate similar to (1.4).

More precisely, let \( A(\rho, R) \subset \mathbb{R}^N \), \( N \geq 2 \), be the annulus with radius \( \rho \) and \( R \) centered at the origin, \( u(x) \) be a radial solution to problem (1.5) in \( \Omega = A(\rho, R) \), and let \( v(r) = u(x) \) for \( r = |x| \). If \( p > 0 \), \( 0 \leq q < (p + 3)/(p + 2) \) and \( p + q > 1 \) they have

\[ v(r) < \phi(R - r)[1 + C(R - r)], \quad r \in (r_1, R), \]  

(1.6)

\[ v(r) > \phi(r - \rho)[1 - C(r - \rho)], \quad r \in (\rho, r_2). \]  

(1.7)

where \( \phi \) be the function defined by

\[ \phi(t) = \left( \frac{2 - q}{p + q - 1} \right)^{\frac{2}{p+q}} \left( \frac{p + 1}{2 - q} \right)^{\frac{1}{p+q}} t^{\frac{2-q}{p+q}}. \]  

(1.8)

and \( r_1 \) is a constant between \( r_0 \) and \( R \), \( r_2 \) is a constant between \( \rho \) and \( r_0 \).

If \( p > 0 \), \( q = (p + 3)/(p + 2) \) they have

\[ v(r) < \phi(R - r)[1 + C(R - r) \ln \frac{1}{R - r}], \quad r \in (r_1, R), \]  

(1.9)
v(r) > \phi(r - \rho)[1 - C(r - \rho)\ln \frac{1}{r - \rho}], \ r \in (\rho, r_2). \quad (1.10)

Let \( \Omega \) be a bounded domain with a smooth boundary \( \partial \Omega \), let \( p > 0, \ 0 \leq q < (p + 3)/(p + 2) \) and \( p + q > 1 \), they have
\[ v(x) \leq u(x) \leq w(x), \]
where \( \phi \) be the function defined in (1.8), and
\begin{align*}
\phi(t) &= (m + 1)^{\frac{1}{m+1}}(p - 1) \left[ \frac{p - 1}{(m + 1) - (p - 1)} \right]^{\frac{1}{m+1}} \left[ \frac{m + 1}{p - 1} \right]^{\frac{1}{p - 1}}. \quad (2.2)
\end{align*}

A solution of (2.1) such that \( \phi(t) \to \infty \) as \( t \to 0 \) is precisely the function defined in (2.2).

In what follows we denote by \( C > 1 \) a constant which may change from term to term.

**Lemma 2.1.** Let \( p > 0, m + 1 > p - 1 \). Consider the equation in (1.1) in dimension \( N = 1 \) and \( \Omega = (0, \infty) \). If \( u = \phi(t) > 0 \) and \( \phi'(t) < 0 \) we have
\[ \phi''(-\phi'p-2) = \phi''(-\phi'). \quad (2.1) \]

where \( \phi(t) \) be defined by
\[ \phi(t) = (m + 1)^{\frac{1}{m+1}}(p - 1) \left[ \frac{p - 1}{(m + 1) - (p - 1)} \right]^{\frac{1}{m+1}} \left[ \frac{m + 1}{p - 1} \right]^{\frac{1}{p - 1}}. \quad (2.2) \]

A solution of (2.1) such that \( \phi(t) \to \infty \) as \( t \to 0 \) is precisely the function defined in (2.2).

**Theorem 2.1.** Let \( A(\rho, R) \subset R^N \), \( N \geq 2 \), be the annulus with radii \( \rho \) and \( R \) centered at the origin. Let \( \phi \) be the function defined in (2.2), let \( u(x) \) be a radial solution to problem (1.1) in \( A(\rho, R) \subset R^N \), and let \( v(r) = u(x) \) for \( r = |x| \). If \( p > 0, \ m + 1 > p - 1 \) we have
\begin{align*}
v(r) &< \phi(R - r)[1 + C(R - r)], \ r \in (r_1, R), \quad (2.3) \\
v(r) &> \phi(r - \rho)[1 - C(r - \rho)], \ r \in (\rho, r_2). \quad (2.4)
\end{align*}

**Proof.** If \( \Omega = A(\rho, R) \), problem 1.1 reads as
\[ \left( |v'|^{p-2}v' \right)' + \frac{N-1}{r}|v'|^{p-2}v' = v^m v', \ v(\rho) = v(R) = \infty. \quad (2.5) \]

There is a point \( r_0 \in (\rho, R) \) such that \( v'(r_0) = 0, \ v'(r) < 0 \) for \( r \in (\rho, r_0) \) and \( v'(r) > 0 \) for \( r \in (r_0, R) \). For \( r \in (r_0, R) \) we have
\[ \left( |v'|^{p-1} \right)' + \frac{N-1}{r}|v'|^{p-1} = v^m v', \ v'(r_0) = 0, \ v(R) = \infty. \quad (2.6) \]
Integration over \((r_0, r)\) yields
\[
(v')^{p-1}|_{r_0}^r + \int_{r_0}^r \frac{N-1}{s} (v')^{p-1} ds = \int_{r_0}^r v^m v' ds,
\]
\[
(v')^{p-1}|_{r_0}^r + (N-1) \int_{r_0}^r \frac{(v')^{p-1}}{s} ds = \frac{v^{m+1} - v_0^{m+1}}{m+1}, \quad v_0 = v(r_0),
\]
\[
(v')^{p-1} + (N-1) \int_{r_0}^r \frac{(v')^{p-1}}{s} ds = \frac{v^{m+1} - v_0^{m+1}}{m+1}.
\] (2.7)

From (2.7) we find
\[
(v')^{p-1} < \frac{v^{m+1}}{m+1},
\]
\[
v' < \frac{v_0^{m+1}}{m+1}.
\]

On the other hand, by lemma 2.2 we have
\[
\lim_{r \to R} \int_{r_0}^r \frac{(v')^{p-1}}{s} ds = \infty,
\]
and combining this with (2.7) implies for \(r \in (r_1, R)\)
\[
2(v')^{p-1} > \frac{v^{m+1}}{m+1}.
\]

Hence by Eq.(2.7) we find
\[
\frac{1}{C} v^{\frac{m+1}{p+1}} < v' < C v^{\frac{m+1}{p+1}}, \quad r \in (r_1, R),
\] (2.8)

From (2.8) we find
\[
\frac{1}{C} v^{\frac{m+1}{p+1}} < \frac{1}{m} < \frac{v^{m+1}}{m+1},
\]
\[
\frac{1}{C} \int_r^R v^{\frac{m+1}{p+1}} v' ds < R - r < C \int_r^R v^{\frac{m+1}{p+1}} v' ds,
\]
\[
\frac{1}{C} \left( 0 - v^{-\frac{m+1}{p+1}} \right) < R - r < C \left( 0 - v^{-\frac{m+1}{p+1}} \right),
\]
\[
\frac{1}{C} v^{-\frac{m+1}{p+1}} < R - r < C v^{-\frac{m+1}{p+1}}.
\]

Finally we get
\[
\frac{1}{C} (R - r)^{-\frac{1}{p+1}} \frac{1}{m} < v < C (R - r)^{-\frac{1}{p+1}} v^{\frac{m+1}{p+1}},
\] (2.9)

and
\[
\frac{1}{C} (R - r)^{-\frac{m+1}{p+1}} v' < C (R - r)^{-\frac{m+1}{p+1}} v^{\frac{m+1}{p+1}}.
\] (2.10)

By using (2.10), we find
\[
\int_{r_0}^r (v')^{p-1} ds < C \int_{r_0}^R \frac{(v^{m+1})^{p-1}}{s} ds
\]
\[
< C^{p-1} \int_{r_0}^R (R - s)^{-\frac{m+1}{p+1}} ds
\]
\[
< C^{p-1} \int_{r_0}^R (R - s)^{-\frac{1}{p+1}} \frac{(m+1)}{(m+1)(p+1)} ds
\]
\[
< C (R - r)^{-\frac{m+1}{p+1}}.
\] (2.11)

Inserting estimate (2.11) into (2.7) we get
\[
(v')^{p-1} > \frac{v^{m+1} - v_0^{m+1}}{m+1} - C (R - r)^{-\frac{m+1+1}{m+1}}.
\]
\[(m + 1) \frac{(v')^{p-1}}{v^{m+1}} > 1 - \frac{C(m + 1)(R - r)}{v^{m+1}} + v_0 v^{p+1}.
\]

From (2.9) we get
\[(m + 1) \frac{(v')^{p-1}}{v^{m+1}} > 1 - C(R - r),
\]
\[(m + 1) \frac{1}{v^{m+1}} v' > 1 - C(R - r).
\]

Integration over \((r, R)\) yields
\[(m + 1) \frac{1}{(p-1) - (m + 1)} \frac{1}{v^{(p-1)-(m+1)}} > R - r - C(R - r)^2,
\]
\[v^{(p-1)-(m+1)} < (m + 1) \frac{1}{(p-1) - (m + 1)} \frac{1}{v^{(p-1)-(m+1)}} (R - r) [1 - C(R - r)],
\]
\[v(r) < (m + 1) \frac{1}{(m+1) - (p-1)} \left[ \frac{1}{(m+1) - (p-1)} \right] (R - r) [1 - C(R - r)] ^{\frac{1}{(m+1) - (p-1)}}.
\]

Since
\[1 - C(R - r) ^{\frac{1}{(m+1) - (p-1)}} < 1 + C(R - r),
\]

with a new constant \(C\), we get
\[v(r) < \phi(r) [1 + C(R - r)],
\]
where \(\phi\) be the function defined by (2.2).

Let us prove inequality (2.4). For \(r \in (\rho, r_0)\) we have \(v'(r) < 0, \)

\[((-v')^{p-2} v')' - \frac{N - 1}{r} (-v')^{p-1} = -v^m v', \quad v(\rho) = \infty, \quad v'(r_0) = 0. \quad (2.12)
\]

Integration over \((r, r_0)\) yields
\[(-v')^{p-2} v' |_{r_0} - (N - 1) \int_{r_0}^{r} \frac{(-v')^{p-1}}{s} ds = -\frac{v^{m+1}}{m+1}, \quad v_0 = v(r_0),
\]
\[0 - (-v')^{p-2} v' - (N - 1) \int_{r}^{r_0} \frac{(-v')^{p-1}}{s} ds = \frac{v^{m+1} - v_0^{m+1}}{m + 1},
\]
\[(-v')^{p-1} - (N - 1) \int_{r}^{r_0} \frac{(-v')^{p-1}}{s} ds = \frac{v^{m+1} - v_0^{m+1}}{m + 1}. \quad (2.13)
\]

Arguing as in the precious case, now we find
\[\frac{1}{C} v^{\frac{m+1}{p-1}} < -v' < C v^{\frac{m+1}{p-1}}, \quad r \in (\rho, r_2),
\]
so
\[\frac{1}{C} (r - \rho)^{\frac{1}{m+1}(p-1)} < v < C (r - \rho)^{\frac{1}{m+1}(p-1)}, \quad (2.14)
\]
and
\[\frac{1}{C} (r - \rho)^{\frac{m+1}{p-1}} < v' < C (r - \rho)^{\frac{m+1}{p-1}}. \quad (2.15)
\]

By using (2.15), we find
\[\int_{r_0}^{r} \frac{(-v')^{p-1}}{s} ds \quad < \int_{r_0}^{r} \frac{C (s - \rho)^{\frac{(m+1)(p-1)}{(m+1)(p-1)-(m+1)}}}{s} ds \quad \frac{(m+2)(p-1)-(m+1)}{(p-1)-(m+1)}
\]
\[< C (r - \rho)^{\frac{(m+2)(p-1)-(m+1)}{(p-1)-(m+1)}}. \quad (2.16)
\]
Inserting estimate (2.16) into (2.13) we get
\[ (-v')^{p-1} < \frac{v'^{m+1}}{m+1} - \frac{v'^{m+1}}{m+1} + C(r - \rho) \frac{(m+2)(p-1) - (m+1)}{(p-1)(m+1)}, \]
\[ (m+1) (-v')^{p-1} - \frac{1}{v'^{m+1}} < 1 + \frac{C(m+1)(r - \rho)}{v'^{m+1}} - \frac{v'^{m+1}}{v'^{m+1}}. \]
From (2.14) we get
\[ (m+1) (-v')^{p-1} - \frac{1}{v'^{m+1}} < 1 + C(r - \rho), \]
\[ (m+1) (-v')^{p-1} - \frac{1}{v'^{m+1}} < 1 + C(r - \rho). \]
Integration over \((\rho, r)\) yields
\[ (m+1) \frac{p - 1}{(m+1) - (p-1)} v^{(p-1)-(m+1)} \leq (r - \rho) + C(r - \rho)^2, \]
\[ (m+1) \frac{p - 1}{(m+1) - (p-1)} [(r - \rho)(1 + C(r - \rho))]^{-1} < v \frac{(m+1) - (p-1)}{p-1}, \]
\[ v(r) > (m+1) \frac{p - 1}{(m+1) - (p-1)} \left[ \frac{p - 1}{(m+1) - (p-1)} \right]^{\frac{p-1}{m+1} - \frac{1-p}{m+1}} (r - \rho)^{\frac{1-p}{m+1}}. \]
Since
\[ (1 + C(r - \rho))^{\frac{1-p}{m+1} - \frac{1-p}{m+1}} > 1 - C(r - \rho), \]
we get
\[ v(r) > \phi(r)[1 - C(r - \rho)]. \]
where \(\phi(r)\) be the function defined by (2.2).

The theorem is proved.

Now let us investigate the problem (1.2). If \(p > 1, 0 < q < 1, \) and \(m + q > p - 1,\) we can get sectional similar arguments as follow.

Let \(p > 1, 0 < q < 1, \) and \(m + q > p - 1,\) Consider the equation in (1.2) in dimension \(N = 1\) and \(\Omega = (0, \infty).\) If \(u = \phi_1(t) > 0\) and \(\phi_1'(t) < 0\) we have
\[ \phi_1''(-\phi_1')^{p-2} = \phi_1''(-\phi_1')^{q}. \]  
(2.17)
Where \(\phi_1\) be defined by
\[ \phi_1(t) = (m+1) \left[ \frac{p - q}{(m+q) - (p-1)} \right]^{\frac{p-1}{m+1} - \frac{1-p}{m+1}} t^{\frac{p-1}{m+1} - \frac{1-p}{m+1}}. \]  
(2.18)

**Theorem 2.2.** Let \(A(\rho, R) \subset R^N, \ N \geq 2,\) be the annulus with radii \(\rho\) and \(R\) centered at the origin. Let \(\phi_1\) be the function defined in (2.17), let \(u(x)\) be a radial solution to problem (1.2) in \(A(\rho, R) \subset R^N,\) and let \(v(r) = u(x)\) for \(r = |x|,\) If \(p > 1, 0 < q < 1, \) and \(m + q > p - 1\) we have
\[ v(r) < \phi_1(R - r)[1 + C(R - r)], \]  
(2.19)
Proof. If $\Omega = A(\rho, R),$ problem (1.1a) reads as

$$\left(r^{N-1} \phi_p (v')\right)' = r^{N-1} v^m |v'|^q, \quad v(\rho) = v(R) = \infty,$$  \hspace{1cm} (2.20)

where $\phi_p (v') = |v'|^{p-2} v'.$ There is a point $r_0 \in (\rho, R)$ such that $v'(r_0) = 0, v'(r) > 0$ for $r \in (r_0, R).$ For $r \in (r_0, R)$ we have

$$\left(r^{N-1} |v'|^{p-2} v'\right)' = r^{N-1} v^m (v')^q.$$

Integration over $(r_0, r)$ yields

$$s^{N-1} \phi_p (v') \bigg|_{r_0}^r = \int_{r_0}^r s^{N-1} v^m (v')^q ds, \quad r \in (r_0, R),$$

$$r^{N-1} \phi_p (v') = \int_{r_0}^r s^{N-1} v^m (v')^q ds,$$

$$\phi_p (v') = \frac{1}{r^{N-1}} \int_{r_0}^r s^{N-1} v^m (v')^q ds,$$

$$v' = \phi_p^{-1} \left( \frac{1}{r^{N-1}} \int_{r_0}^r s^{N-1} v^m (v')^q ds \right).$$

where

$$\phi_p^{-1} (s) = \begin{cases} \frac{s^{p-1}}{p-1}, & s \geq 0 \\ (-s)^{p-1}, & s < 0. \end{cases}$$

We get

$$v' = \left( \frac{1}{r^{N-1}} \int_{r_0}^r s^{N-1} v^m (v')^q ds \right)^{\frac{1}{p-1}}.$$  \hspace{1cm} (2.21)

Since $r \in (r_0, R),$ we find

$$\int_{r_0}^r s^{N-1} v^m (v')^q ds \leq R^{N-1} \int_{r_0}^r v^m (v')^q ds.$$  \hspace{1cm} (2.22)

From Hölder inequality we get

$$\int_{r_0}^r v^m (v')^q ds \leq \left( \int_{r_0}^r (v^m (v')^q) \frac{1}{q} ds \right)^q |r - r_0|^{1-q} = \left( \int_{r_0}^r v^{\frac{m}{q}} v' ds \right)^q |r - r_0|^{1-q}.$$  \hspace{1cm} (2.22)

From (2.22) we get

$$v' < \left[ \frac{R^{N-1}}{r^{N-1}} \left( \int_{r_0}^r v^{\frac{m}{q}} v' ds \right)^q (R - r_0)^{1-q} \right]^{\frac{1}{p-1}},$$

$$v' < \left[ \frac{R^{N-1} (R - r_0)^{1-q}}{r^{N-1}} \right]^{\frac{1}{p-1}} \left[ \left( \int_{r_0}^r v^{\frac{m}{q}} v' ds \right)^q \right]^{\frac{1}{p-1}},$$

$$v' < \left[ \frac{r^{N-1} (R - r_0)^{1-q}}{r^{N-1}} \right]^{\frac{1}{p-1}} \left[ \left( \int_{r_0}^r v^{\frac{m}{q}} v' ds \right)^q \right]^{\frac{1}{p-1}},$$

$$v' < C \left[ \frac{m + q}{m + q - m + q} \right]^{\frac{1}{p-1}},$$  \hspace{1cm} (2.23)

we get

$$v' < C v^{\frac{m+q}{m+q-q}} < C v^{\frac{m}{p-1}}.$$  \hspace{1cm} (2.23)
By using (2.23) we get

\[ v < C(R - r)^{\frac{q-p}{(p-1)-(m+q)}}, \tag{2.24} \]

and

\[ v' < C(R - r)^{\frac{m+1}{(p-1)-(m+q)}}. \tag{2.25} \]

While, the problem (1.2) reads as

\[ (\phi_p(v'))' + \frac{N-1}{r}(v')^{p-1} = v^m(v')^q. \tag{2.26} \]

From (2.26) we find

\[ (v')^{1-q}(\phi_p(v'))' + \frac{N-1}{r}(v')^{p-q} = v^m v', \tag{2.27} \]

integration for \( r \) we get

\[
\int_{r_0}^r (v')^{1-q}(\phi_p(v'))' \, ds + \int_{r_0}^r \frac{N-1}{s}(v')^{p-q} \, ds = \int_{r_0}^r v^m v' \, ds,
\]

\[
(v')^{p-q} + \frac{N-1}{s} (v')^{p-q} \, ds = \frac{v^{m+1} - v_0^{m+1}}{m+1} + \int_{r_0}^r \phi_p(v') \left((v')^{1-q}\right)' \, ds,
\]

\[
(v')^{p-q} + (N-1) \int_{r_0}^r \frac{(v')^{p-q}}{s} \, ds > \frac{v^{m+1} - v_0^{m+1}}{m+1}. \tag{2.28}
\]

Since \( 0 < q < 1 \), by (2.27)

\[
\int_{r_0}^r \frac{(v')^{p-q}}{s} \, ds < C(R - r)^{\frac{(m+2)(p-q)-(m+1)}{(p-1)-(m+q)}}.
\]

From (2.28) we get

\[
(v')^{p-q} > \frac{v^{m+1} - v_0^{m+1}}{m+1} - C(N-1)(R - r)^{\frac{(m+2)(p-q)-(m+1)}{(p-1)-(m+q)}},
\]

\[
(m+1) \frac{(v')^{p-q}}{v^{m+1}} > 1 - C(R - r),
\]

\[
(m+1)^\frac{q-p}{(m+q)-(p-1)} \frac{v'}{v^{\frac{p-q}{(m+q)-(p-1)}}} > 1 - C(R - r).
\]

Integration for \( r \) we get

\[
(m+1)^\frac{q-p}{(m+q)-(p-1)} \left[ \frac{v^{(m+q)-(p-1)}}{v^{p-q}} \right]_{r_0}^r > (R - r) - C(R - r)^2,
\]

\[
v < (m+1)^{\frac{1}{(m+q)-(p-1)}} \left[ \frac{p-q}{(m+q)-(p-1)} \right]_{r_0}^r \left[ (R-r)^{\frac{q-p}{(m+q)-(p-1)}} \frac{(1-C(R-r))^{\frac{q-p}{(m+q)-(p-1)}}}{1-C(R-r)} \right]_{r_0}^r.
\]

Since

\[
[1-C(R-r)]^{\frac{q-p}{(m+q)-(p-1)}} < 1 + C(R - r),
\]

we get

\[ v(r) < \phi_1(r)[1 + C(r - r)], \]

where \( \phi_1(r) \) be the function defined by (2.18).

The theorem is proved.
3 Estimates for Boundary Blowup Solution

In this section we study the estimate for boundary blowup solution of problem (1.1) and (1.2).

Lemma 3.1. Let $\Omega \in \mathbb{R}^N$, $N \geq 2$, be a bounded domain satisfying an interior and an exterior sphere condition at each point of its boundary $\partial \Omega$. Let $\phi$ be the function introduced in (2.2), let $u(x)$ be a solution to problem (1.1) in $\Omega$, and let $\delta = \delta(x)$ be the distance from $x$ to $\partial \Omega$. If $p > 1$, and $m + 1 > p - 1$ we have

$$\phi(\delta)(1 - C\delta) < u(x) < \phi(\delta)(1 + C\delta). \quad (3.1)$$

Proof. The proof uses theorem 2.1 and the comparison principle for elliptic equation (see for example [15, Theorem 10.1]).

Theorem 3.1. Let $\Omega$ be a bounded domain with a smooth boundary $\partial \Omega$, let $\phi$ be the function introduced in (2.2), and let $\delta = \delta(x)$ be the distance from $x$ to $\partial \Omega$. Let $p > 1$, and $m + 1 > p - 1$. Define

$$w(x) = \phi(\delta) \left(1 + \frac{(p - 1)(N - 1)H(x)}{2(m + 2)(p - 1) - (m + 1)} \delta + \alpha \delta^\sigma\right), \quad (3.2)$$

where $H(x)$ denotes the mean curvature of the surface ($\delta(x) = \text{constant}$) at the point $x$. If $u$ is a solution to problem (1.1), $\sigma > 1$ is a suitable number and $\alpha$ is large enough then

$$u(x) \leq w(x).$$

Furthermore, if

$$v(x) = \phi(\delta) \left(1 + \frac{(p - 1)(N - 1)H(x)}{2(m + 2)(p - 1) - (m + 1)} \delta - \alpha \delta^\sigma\right), \quad (3.3)$$

then

$$v(x) \leq u(x).$$

Proof. From (2.2) we find

$$\frac{\phi(t)}{-\phi'(t)} = \frac{(m + 1) - (p - 1)t}{m - 1},$$

$$\frac{-\phi'(t)}{\phi''(t)} = \frac{(m + 1) - (p - 1)}{m + 1}t,$$

$$\frac{\phi(t)}{\phi''(t)} = \frac{[(m + 1) - (p - 1)]^2}{(p - 1)(m + 1)}t^2. \quad (3.4)$$

Let $K = (N - 1)H$ and

$$A = \frac{(p - 1)K}{2((m + 2)(p - 4) - (m + 1))}. \quad (3.5)$$

Then

$$w = \phi(\delta)(1 + A\delta + \alpha \delta^\sigma). \quad (3.6)$$

We have

$$\nabla w = \phi'\nabla w(1 + A\delta + \alpha \delta^\sigma) + \phi(\nabla A\delta + A\nabla \delta + \alpha \sigma \delta^{\sigma - 1}\nabla \delta). \quad (3.7)$$

Since (see for example [10])

$$|\nabla \delta| = 1, \quad \Delta \delta = -(N - 1)H = -K,$$

we find

$$\Delta w = \phi''\nabla \delta \nabla \delta + \phi'\Delta \delta(1 + A\delta + \alpha \delta^\sigma) + \phi'\nabla \delta(\nabla A\delta + A\nabla \delta + \alpha \sigma \delta^{\sigma - 1}\nabla \delta) + \phi(\Delta A\delta + \nabla A\nabla \delta + \nabla A\nabla \delta + \alpha \Delta \delta + \alpha \sigma (\sigma - 1)\delta^{\sigma - 2}\nabla \delta + \alpha \sigma \delta^{\sigma - 1}\Delta \delta).
$$

$$= \phi'\left(\phi''(K)(1 + A\delta + \alpha \delta^\sigma) + 2\phi(\nabla A\nabla \delta + A + \alpha \sigma \delta^{\sigma - 1}) + \phi(\Delta A\delta + 2\nabla A\nabla \delta - AK + \alpha \sigma (\sigma - 1)\delta^{\sigma - 2} - \alpha \sigma \delta^{\sigma - 1}K).$$
By using (3.4) we find
\[
\Delta w = \phi' \left[ \left( 1 + \frac{(m+1)-(p-1)}{p-1} A\delta \right) + \frac{(m+1)-(p-1)}{p-1} \delta (\nabla^2 \nabla\delta + A + \alpha\sigma\delta^2) \right] + \frac{(m+1)-(p-1)}{(p-1)(m+1)} \delta^2 (\Delta A\delta + 2\nabla A\nabla\delta - AK + \alpha\sigma(\sigma - 1)\delta^2 - \alpha\sigma\delta^2 K) ,
\]
we get
\[
\Delta w = \phi'' \left[ 1 + A\delta + \frac{(m+1)-(p-1)}{m+1} \delta (K - 2A) + O(1)\delta^2 \right] + \alpha\sigma \left( \phi (1 - 2\sigma \left( \frac{(m+1)-(p-1)}{p-1} \right) + O(1)\delta) \right),
\]
where \( O(1) \) denotes a bounded quantity as \( \delta \to 0 \).

Now we estimate \( |\nabla w| \).
\[
\nabla w = \phi' \nabla w(1 + A\delta + \alpha\delta^2) + \phi(\nabla A\delta + A\nabla\delta + \alpha\sigma\delta^2 \nabla\delta)
\]
\[
= \phi' \left[ \nabla \delta (1 + A\delta + \alpha\delta^2) - \frac{(m+1)-(p-1)}{p-1} \delta (\nabla A\delta + A\nabla\delta + \alpha\sigma\delta^2 \nabla\delta) \right]
\]
\[
= \phi' \left[ \nabla \delta \left( 1 + A \frac{(p-1)-(m+1)}{p-1} \delta + \alpha\sigma (1 - \frac{(m+1)-(p-1)}{p-1}) \sigma \right) - \frac{(m+1)-(p-1)}{p-1} \nabla A\delta^2 \right].
\]

Fix \( \alpha \) and \( \sigma \), we take \( \delta \) so small that
\[
1 + A \frac{(p-1)-(m+1)}{p-1} \delta + \alpha\sigma (1 - \frac{(m+1)-(p-1)}{p-1}) \sigma > 0.
\]
Then, we have
\[
|\nabla w| = (\phi') \left[ 1 + A \frac{(p-1)-(m+1)}{p-1} \delta + \alpha\sigma (1 - \frac{(m+1)-(p-1)}{p-1}) \sigma + O(1)\delta^2 \right].
\]

and
\[
|\nabla w|^{p-2} = (\phi')^{p-2} \left[ 1 + A \frac{(p-1)-(m+1)}{p-1} \delta + \alpha\sigma (1 - \frac{(m+1)-(p-1)}{p-1}) \sigma + O(1)\delta^2 \right]^{p-2}
\]
\[
+ O(1)\delta^2 + O(1)(\alpha\sigma^2)^2 \right].
\]

By using (3.8) we get
\[
|\nabla w|^{p-2} \Delta w = (\phi')^{p-2} \phi'' \left[ 1 + A\delta + \frac{(m+1)-(p-1)}{m+1} \delta (K - 2A) \right]
\]
\[
+ A(p-2) \frac{(p-1)-(m+1)}{p-1} \delta + O(1)\delta^2 \right]
\]
\[
+ (\phi')^{p-2} \phi''(\alpha\sigma^2) \left[ 1 - 2\sigma \frac{(m+1)-(p-1)}{m+1} \sigma + O(1)(\sigma\delta^2) \right]
\]
\[
+ (p-2)(1 - \frac{(m+1)-(p-1)}{p-1}) \sigma + O(1)\delta^2 + O(1)(\alpha\sigma^2)^2 \right].
\]

Let us estimate \( w^m \). We have
\[
w^m = \phi^m (1 + A\delta + \alpha\delta^2)^m
\]
\[
= \phi^m \left[ 1 + mA\delta + m\alpha\delta^2 + m(m+1)(1+\omega)^{m+2} \frac{(A\delta + \alpha\delta^2)^2}{2} \right].
\]

Where \( \omega \) is a quantity in between 0 and \( A\delta + \alpha\delta^2 \). From now on, we choose \( \alpha \), \( \sigma \) and \( \rho \) such that
\[
-\frac{1}{2} \leq A\delta + \alpha\delta^2 \leq 1.
\]

Then \( \frac{1}{2} < 1 + \omega < 2 \), and
\[
w^m = \phi^m \left[ 1 + mA\delta + m\alpha\delta^2 + O(1)\delta^2 + O(1)(\alpha\delta^2)^2 \right].
\]
Since \( \phi''(-\phi')^{p-2} = \phi^m(-\phi') \), by (3.9) and (3.11) we find
\[
w^m|\nabla w| = \phi''(-\phi')^{p-2} \left[ 1 + A \left( m + \frac{2(p-1)-(m+1)}{p-1} \right) \delta + \alpha \delta^\sigma \left( m + 1 - \frac{(m+1)-(p-1)}{p-1} \sigma \right) + O(1) \right]^{\delta^2} + O(1) \left( \alpha \delta^\sigma \right)^2
\]
(3.12)

Using (3.10) and (3.12), the inequality
\[
\text{div} \left( |\nabla w|^{p-2} \nabla w \right) < w^m|\nabla w|
\]
reads as
\[
(-\phi')^{p-2} \phi'' \left[ 1 + A \delta + \frac{(m+1)-(p-1)}{m+1} \delta (K - 2A) + A(p-2) \frac{2(p-1)-(m+1)}{p-1} \delta + O(1) \right]^{\delta^2} \\
+(-\phi')^{p-2} \phi'' \left[ 1 - 2\sigma \frac{(m+1)-(p-1)}{m+1} + \sigma(\sigma - 1) \frac{(m+1)-(p-1)^2}{p-1} \left( \frac{m+1}{m+1} \right) \right]^{\delta^2} \\
+(p-2)(1 - \frac{(m+1)-(p-1)}{p-1} \sigma) + O(1) \delta^2 + O(1) \left( \alpha \delta^\sigma \right)^2
\]
(3.13)

We claim that
\[
A + \frac{(m+1)-(p-1)}{m+1} (K - 2A) + A(p-2) \frac{2(p-1)-(m+1)}{p-1} = A \left( m + \frac{2(p-1)-(m+1)}{p-1} \right).
\]

Indeed, we have
\[
\frac{(m+1)-(p-1)}{m+1} (K - 2A) = A \left( \frac{(m+1)(p-2)}{p-1} - A \frac{2(p-1)-(m+1)}{p-1} \right) = 2A \frac{p-1}{p-1} ((m+1) - (p-1)),
\]
then we get
\[
K - 2A = 2A \frac{(m+1)(p-2)}{p-1},
\]
and
\[
K = 2A \frac{(m+2)(p-1) - (m+1)}{p-1}.
\]

The latter equation follows easily from (3.5). Hence, inequality (3.13) holds provided
\[
C_1 \delta^2 + \alpha \delta^\sigma \left( 1 - 2\sigma \frac{(m+1)-(p-1)}{m+1} + \sigma(\sigma - 1) \frac{(m+1)-(p-1)^2}{p-1} \left( \frac{m+1}{m+1} \right) \right) + (p-2) \left( 1 - \frac{(m+1)-(p-1)}{p-1} \right) \sigma + C_2 \delta + C_3 \alpha \delta^\sigma
\]
where \( C_1, C_2 \) and \( C_3 \) are suitable constant. After simplification we find
\[
C_4 \delta^2 \leq \alpha \delta^\sigma \left( (m+1) - (p-1) \right) \left( 1 - \frac{p-3}{p-1} \sigma + \frac{2}{m+1} \sigma \right) - \sigma(\sigma - 1) \frac{(m+1)-(p-1)}{p-1} \left( \frac{m+1}{m+1} \right) \left( C_2 \delta + C_3 \alpha \delta^\sigma \right),
\]
(3.14)
The quantity
\[
1 - \frac{p-3}{p-1} \sigma + \frac{2}{m+1} \sigma - \sigma(\sigma - 1) \frac{(m+1)-(p-1)}{p-1} \left( \frac{m+1}{m+1} \right)
\]
computed at \( \sigma = 1 \) becomes
\[
\frac{2(m+1) + (p-1)}{(m+1)(p-1)}.
\]
Which is positive. By continuity, we have
\[
1 - \frac{p - 3}{p - 1} \sigma + \frac{2}{m + 1} \sigma - \sigma(\sigma - 1) \frac{(m + 1) - (p - 1)}{(p - 1)(m + 1)} > 0,
\]
with a suitable \( \sigma > 1 \). Fixed such a value of \( \sigma \), choose \( \alpha \) and \( \delta \) so that
\[
1 - \frac{p - 3}{p - 1} \sigma + \frac{2}{m + 1} \sigma - \sigma(\sigma - 1) \frac{(m + 1) - (p - 1)}{(p - 1)(m + 1)} - C_2 \delta + C_3 \alpha \delta^\sigma > 0.
\]
The inequality (3.13) (and the inequality \( \text{div} \left( |\nabla w|^{p-2} \nabla w \right) < w^m |\nabla w| \)) holds for \( \alpha \) large enough and \( x \) such that \( \delta(x) \leq \delta_0 \), with a suitable \( \delta_0 \).

Consider the domain \( \Omega_{\delta_1} = \{ x \in \Omega, \ \delta(x) < \delta_0 \} \). Let us show that, for \( \delta_1 \) small enough, \( u(x) \leq w(x) \) on \( \Omega_{\delta_1} \). Indeed, by lemma 3.1, we know that
\[
w(x) < \phi(\delta)(1 + C\delta).
\]
Hence,
\[
w(x) - u(x) > \phi(\delta)(1 + A\delta + \alpha \delta^\sigma) - \phi(\delta)(1 + C\delta) = \phi(\delta)((A - C)\delta + \alpha \delta^\sigma).
\]
Let \( \alpha_\theta \) and \( \delta_0 \) such the inequality (3.13) holds for \( \delta \leq \delta_0 \). Decrease \( \delta \) (increasing \( \alpha \)) so that \( \alpha_1 \delta_1^\sigma = \alpha_0 \delta_0^\sigma \) until
\[
(A - C)\delta_1 + \alpha_1 \delta_1^\sigma > 0.
\]
Then \( w(x) \geq u(x) \) for \( \delta(x) = \delta_1 \).

Now we introduce a number \( 0 < \theta < 1 \), of course, we have \( w(x) > \theta u(x) \) for \( x \) such that \( \delta(x) = \delta_1 \). On the other hand, using lemma 3.1 again we have
\[
w(x) - \theta u(x) > \phi(\delta)(1 - \theta + (A - C\theta)\delta + \alpha \delta^\sigma).
\]
As \( \delta \to 0 \) (with \( \alpha \) fixed) we have
\[
1 - \theta + (A - C\theta)\delta + \alpha \delta^\sigma > 0.
\]
Hence, \( w(x) - \theta u(x) > 0 \) near \( \partial \Omega_\theta \).

Since \( 0 < \theta < 1 \) and \( m + 1 - (p - 1) > 0 \), by (1.1) we find
\[
\text{div} \left( |\nabla(\theta u)|^{p-2} \nabla(\theta u) \right) > (\theta u)^m |\nabla(\theta u)|
\]
(3.15)
Indeed, since
\[
\Delta_p(u) = u^m |\nabla u|,
\]
we find
\[
\Delta_p(\theta u) = \theta^{p-1} \Delta_p u,
\]
and
\[
(\theta u)^m |\nabla(\theta u)| = \theta^{m+1} u^m |\nabla u|,
\]
then we get
\[
\Delta_p(\theta u)/(\theta u)^m |\nabla(\theta u)| = \theta^{p-1-(m+1)} > 1.
\]
The (3.15), together with the inequality \( \text{div} \left( |\nabla w|^{p-2} \nabla w \right) < w^m |\nabla w| \), and the condition \( \theta u(x) \leq w(x) \) on \( \partial \Omega_{\delta_1} \), imply that \( \theta u(x) \leq w(x) \) on \( \Omega_{\delta_1} \). As \( \theta \to 1 \), we find \( u(x) \leq w(x) \) on \( \Omega_{\delta_1} \). Increasing \( \alpha \) we get \( u(x) \leq w(x) \) on \( \Omega \). The first assertion of the theorem follows.
To get the inequality \( v(x) \leq u(x) \). We adopt a similar argument. To place of (3.10) we find, with
\( v = \phi(\delta)(1 + A\delta - \alpha\delta^\sigma) \), where \( A \) is as in (3.5),
\[
|\nabla v|^p - 2 \Delta v = (-\phi')^{p-2}\phi''[1 + A\delta + \frac{(m+1)-(p-1)}{m+1}\delta(K - 2A) + A(p - 2)\frac{2(p-1)-(m+1)}{p-1}\delta + O(1)\delta^2]
\]
\[
+ O(1)(\sigma - 1)(\frac{(m+1)-(p-1)}{p-1}\sigma + O(1)\alpha\delta^\sigma)
\]
\[
+ (p - 2)(1 - \frac{(m+1)-(p-1)}{p-1}\sigma + O(1)\alpha\delta^\sigma)^2) + o(\delta^2)
\] (3.16)

In place of (3.12), we have
\[
v^m|\nabla v| = \phi'''(-\phi')^{p-2}[1 + A \left( m + \frac{2(p-1)-(m+1)}{p-1}\delta - \alpha\delta^\sigma \left( m + 1 - \frac{(m+1)-(p-1)}{p-1}\sigma \right) \right)
\]
\[
+ O(1)\delta^2 + O(1)(\alpha\delta)^2].
\] (3.17)

Using (3.16) and (3.17), the inequality
\[
\text{div} \left( |\nabla v|^p - 2 \nabla v \right) > v^m|\nabla v|
\] (3.18)
reads as
\[
(-\phi')^{p-2}\phi'' \left[ 1 + A\delta + \frac{(m+1)-(p-1)}{m+1}\delta(K - 2A) + A(p - 2)\frac{2(p-1)-(m+1)}{p-1}\delta + O(1)\delta^2 \right]
\]
\[
- (-\phi')^{p-2}\phi''(\alpha\delta^\sigma) \left( 1 - 2\sigma\frac{(m+1)-(p-1)}{m+1} + \sigma(\sigma - 1)(\frac{(m+1)-(p-1)}{p-1}\sigma + O(1)\delta^2 + O(1)(\alpha\delta^\sigma)^2 \right)
\]
\[
+ (p - 2)(1 - \frac{(m+1)-(p-1)}{p-1}\sigma + O(1)\alpha\delta^\sigma)^2) \right)
\] (3.19)

After simplification we find
\[
-C_1\delta^2 - \alpha\delta^\sigma(1 - 2\sigma\frac{(m+1)-(p-1)}{m+1} + \sigma(\sigma - 1)(\frac{(m+1)-(p-1)}{p-1}\sigma + (p - 2)(1 - \frac{(m+1)-(p-1)}{p-1}\sigma) + 2\sigma - C_2\delta - C_3\alpha\delta^\sigma),
\] (3.20)
which is equivalent to (3.14). Hence, we have \( \text{div} \left( |\nabla v|^p - 2 \nabla v \right) > v^m|\nabla v| \) for large enough and \( x \) such that \( \delta(x) \leq \delta_0 \), \( u(x) \geq v(x) \) on \( \Omega_{\delta_1} \). Indeed, by lemma 3.1 we know that
\[
u(x) > \phi(\delta)(1 - C\delta).
\]

Hence,
\[
u(x) - u(x) < \phi(\delta)((A + C)\delta - \alpha\delta^\sigma).
\]

Let \( \alpha_0 \) and \( \delta_0 \) such that inequality (3.20) holds for \( \delta \leq \delta_0 \). Decrease \( \delta \) (increasing \( \alpha \) so that \( \alpha_1\delta^\sigma = \alpha_0\delta^\sigma \)) until
\[
(A + C)\delta_1 - \alpha_2\delta_1^\sigma < 0.
\]

Then \( u(x) \geq v(x) \) for \( \delta(x) = \delta_1 \).

Now, for \( \Theta > 1 \) we have \( v(x) < \Theta u(x) \) for \( x \) such that \( \delta(x) > \delta_1 \). On the other hand, by lemma 2.2 it follows that \( v(x) \leq \Theta u(x) \) for \( x \) near \( \partial\Omega \). We have proved that proved that \( v(x) \leq \Theta u(x) \) on \( \partial\Omega_{\delta_1} \). Since \( \Theta > 1 \) and \( m + 1 - (p - 1) > 0 \), by (1.1a) we find
\[
\Delta_p(\Theta u) < (\Theta u)^m|\nabla(\Theta u)|.
\]
The latter inequality, together with the inequality (3.18) and the condition \( v(x) \leq \Theta u(x) \) on \( \partial\Omega_{\delta_1} \), imply that \( v(x) \leq \Theta u(x) \) on \( \Omega_{\delta_1} \). As \( \Theta \to 1 \) we find \( v(x) \leq u(x) \) on \( \Omega_{\delta_1} \). Increasing \( \alpha \) we get \( v(x) \leq u(x) \) on \( \Omega \).

The theorem is proved.
Now, when \( p > 0 < q < 1 \), and \( m + q > p - 1 \), we get partial argument similar to Theorem 3.1.

**Lemma 3.2.** Similar to lemma 3.1, \( \phi_{1} \) be the function introduced in (2.18), let \( u(x) \) be a solution to problem (1.1a) in \( \Omega \). If \( p > 0 < q < 1 \), and \( m + q > p - 1 \), we have

\[
u(x) < \phi(\delta)(1 + C\delta).
\]

**Theorem 3.2.** Let \( \Omega \) be a bounded domain with a smooth boundary \( \partial\Omega \), let \( \phi \) be the function introduced in (2.18), and let \( \delta = \delta(x) \) be the distance from \( x \) to \( \partial\Omega \). Let \( p > 1, \ 0 < q < 1 \), and \( m + 1 > p - 1 \). Define

\[
w(x) = \phi(\delta) \left( 1 + \frac{(p - q)(N - 1)H(x)}{2((m + 2)(p - q) - (m + 1))}\delta + \alpha\delta^\alpha \right),
\]

where \( H(x) \) denotes the mean curvature of the surface \( (\delta(x) = \text{constant}) \) at the point \( x \). If \( u \) is a solution to problem (1.2), \( \sigma > 1 \) is a suitable number and \( \alpha \) is large enough then

\[
u(x) \leq w(x).
\]

**Proof.** From (2.18) we find

\[
\begin{align*}
\phi(t) &= (m + q - (p - 1)t, \\
-\phi'(t) &= (m + q - (p - 1)t), \\
-\phi''(t) &= \frac{m + q - (p - 1)}{m + 1}, \\
\phi(t) \phi''(t) &= \left[(m + q - (p - 1))^2\right] \left[(p - q)(m + 1)\right].
\end{align*}
\]

(3.21)

Let \( K = (N - 1)H \) and

\[
A = \frac{(p - q)K}{2((m + 2)(p - q) - (m + 1))},
\]

then

\[
w = \phi(\delta)(1 + A\delta + \alpha\delta^\alpha).
\]

In place of (3.8) we have

\[
\Delta w = \phi'' \left[ 1 + A\delta + \frac{(m + q - (p - 1))}{m + 1}\delta(K - 2A) + O(1)\delta^2 + \alpha\delta^\alpha \right] \left[ 1 - 2\sigma \frac{(m + q - (p - 1))}{m + 1} + \sigma(\sigma - 1) \frac{(m + q - (p - 1))^2}{(p - q)(m + 1)} + O(1)\delta \right].
\]

(3.23)

Then we get the estimate for \( |\nabla w| \),

\[
|\nabla w| = (-\phi') \left[ 1 + A\delta + \frac{(m + q - (p - 1))}{m + 1}\delta(K - 2A) + O(1)\delta^2 \right] \left[ 1 - \frac{(m + q - (p - 1))}{p - q} + O(1)\delta^2 \right].
\]

(3.24)

In place of (3.10) we get

\[
|\nabla w|^{p - 2} \Delta w = (-\phi')^{p - 2} \phi'' \left[ 1 + \frac{(m + q - (p - 1))}{m + 1}\delta(K - 2A) + O(1)\delta^2 \right] \left[ 1 - \frac{(m + q - (p - 1))}{p - q} + O(1)\delta^2 \right] \left[ 1 - \frac{(m + q - (p - 1))^2}{(p - q)(m + 1)} \right] \left[ 1 - \frac{(m + q - (p - 1))^2}{p - 1}\delta \right].
\]

(3.25)

Let us estimate \( |\nabla w|^q \). By using (3.9) we get

\[
|\nabla w|^q = (-\phi')^q \left[ 1 + A\delta + \frac{(m + q - (p - 1))}{m + 1}\delta + \alpha\delta^\alpha \right] \left[ 1 - \frac{(m + q - (p - 1))}{p - q} + O(1)\delta^2 \right]^q \left[ 1 + q\alpha\delta^\alpha \left[ 1 - \frac{(m + q - (p - 1))}{p - q} \right] + O(1)\delta^2 \right].
\]

(3.26)
By using (3.11) and (3.26) we have

\[ w^m |\nabla w|^q = \phi''(-\phi')^q \left[ 1 + A \left( m + q \frac{(p-q)+(p-1)-(m+q)}{p-q} \right) + \alpha \delta^\sigma \left( m + q - \frac{q}{p-q} \alpha \delta \right) + O(1) \delta^2 + O(1)(\alpha \delta^\sigma)^2 \right]. \tag{3.27} \]

By (3.25) and (3.27), the inequality

\[ \text{div}(|\nabla w|^{p-2}\nabla w) < w^p |\nabla w|^q \]
reads as

\[
\begin{align*}
&(-\phi')^{p-2} \phi'' \left[ 1 + A\delta + \frac{(m+q)-(p-1)}{m+1} \delta[K-2A] + A(p-2) \frac{(p-q)+(p-1)-(m+q)}{p-q} \delta + O(1) \delta^2 \right] \\
&\quad + (-\phi')^{p-2} \phi''(\alpha \delta^\sigma) \left( 1 - 2\sigma \frac{(m+q)-(p-1)}{m+1} + \sigma(\sigma-1) \frac{(m+q)-(p-1)^2}{(p-q)(m+q)} \right) \\
&\quad + (p-2) \left( 1 - \frac{(m+q)-(p-1)}{p-1} \right) \delta + O(1) \delta^2 + O(1)(\alpha \delta^\sigma)^2 \tag{3.28} \end{align*}
\]

We claim that

\[ A + \frac{(m+q)-(p-1)}{m+1} (K-2A) + A(p-2) \frac{(p-q)+(p-1)-(m+q)}{p-q} = A \left( m + q \frac{p-q}(p-q) \right). \]

Indeed, we have

\[
\begin{align*}
\frac{(m+q)-(p-1)}{m+1} (K-2A) &= \frac{2(p-q-1)(m+q)-(p-1)}{p-q}, \\
K &= 2A + 2 \frac{(m+1)(p-q-1)}{p-q}, \\
K &= 2 \frac{(m+2)(p-q)-(m+1)}{p-q}.
\end{align*}
\]

The latter equation follows easily from (3.22). Hence, (3.28) holds provided

\[
C_1 \delta^2 + \alpha \delta^\sigma \left( 1 - 2\sigma \frac{(m+q)-(p-1)}{m+1} + \sigma(\sigma-1) \frac{(m+q)-(p-1)^2}{(p-1)(m+q)} + (p-2) \left( 1 - \frac{(m+q)-(p-1)}{p-1} \right) \right) < \alpha \delta^\sigma \left( m + q - \frac{q}{p-q} \alpha \delta \right) - C_2 \delta + C_3 \alpha \delta^\sigma,
\]

where \(C_1, C_2,\) and \(C_3\) are suitable constants. After simplification we find

\[
C_1 \delta^2 \leq \alpha \delta^\sigma \left( (m+q) - (p-1) \right) \left( 1 - \frac{2(p-1)}{p-q} \sigma \right) - C_2 \delta + C_3 \alpha \delta^\sigma. \tag{3.29}
\]

Which is equivalent to (3.14). Hence, we have

\[ \text{div}(|\nabla w|^{p-2}\nabla w) < w^p |\nabla w|^q \]
for a large enough and \(x\) such that \(\delta(x) \leq \delta_0\), with a suitable \(\delta_0\). Arguing as in the proof of the previous theorem one prove that \(w(x) \geq u(x)\) in \(\Omega\).
4 Conclusion

We introduce the concept of the boundary blowup solutions of \( p \)-Laplacian type quasilinear elliptic equations. We obtain that the estimate of the radial solution in the annulus, and that the estimate of the boundary blowup solution on a bounded domain.

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Competing Interests

The authors declare that no competing interests exist.

References


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