On the Monotonic Solutions of Quadratic Integral Equations in Orlicz Space

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Authors’ contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

In this paper, Darbo’s and Rothe’s fixed point theorems is used to prove the existence of monotonic solutions for the nonlinear quadratic fractional order integral equation of the following type:

\[ x(t) = h(t) + Gx(t) \int_{a}^{t} f(H_1 x(t), H_2 x(t)) \, dt, \quad t \in [1, e], \quad \alpha > 0, \]

where \( a \) belongs to appropriate Orlicz space. Here \( J^\alpha \) stands the Hadamard-type fractional integral operator.

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1 Introduction and Preliminaries

Prompted by the application of the quadratic functional integral equations to nuclear physics, this equations have provoked some interest in the literature (cf. [1], [2]-[6] and [7]). Specifically, the so-called quadratic integral equations of Chandrasekher type can be very often encountered in many applications (cf. [1], [8] and [9]). Some problems in the queuing theory and biology lead to the quadratic functional integral equation of fractional type (cf. e.g. [10] and [11])

\[ x(t) = h(t) + Gx(t)\mathcal{J}^\alpha a(t)f(\max_{s \in [t,1]} |x(s)|, \int_1^t |x(s)| \, ds), \quad t \in [1, e], \quad \alpha > 0, \quad (1) \]

The aim of this paper is to prove the existence of monotonic solutions to the quadratic functional integral equation of type (1) in the Banach space \( C[1, e] \). Our results in this paper are motivated by the extensions of the work of Banas and Martinon (cf. [2] and [5]) based on the a measure of noncompactness and fixed point theorem due to Darbo and construct an example that proves the existence of the solution in the Banach space \( C[1, e] \) but does not apply the conditions in (cf. [4], [12] and [13]).

Let \( L_p[1, e] \) (1 ≤ p ≤ ∞) denotes the Banach space of \( p \) integrable functions on the interval \([1, e]\) endowed standard norm \( ||\cdot||_p \) while \( C[1, e] \) be the space of continuous functions on the interval \([1, e]\) with the usual sup-norm. Recall that the Hadamard fractional integral operator of order \( \alpha > 0 \) with left-hand point 1 is defined by

\[ \mathcal{J}^\alpha x(t) := \frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha-1} \frac{x(s)}{s} \, ds, \quad t > 1. \quad (2) \]

\( \mathcal{J}^\alpha \) may indeed be considered as a corresponding fractional integral. For more on Hadamard fractional operators, we refer the interested reader to see (e.g. [14], [15] and [16] and the references therein). Let \( \psi: \mathbb{R}^+ \to \mathbb{R}^+ \) be a Young-type function, (i.e. \( \psi \) is increasing, even, convex and continuous with \( \psi(0) = 0 \) and \( \lim_{u \to \infty} \psi(u) = \infty \)). The Orlicz space \( L_\psi = L_\psi([a, b], \mathbb{R}) \) is the Banach space consists of all measurable function \( x: [a, b] \to \mathbb{R} \) for which there exists a number \( k > 0 \) such that

\[ \int_a^b \psi \left( \frac{|x(s)|}{k} \right) \, ds < 1. \quad (3) \]

with (Luxemburg) norm \( \|x\|_\psi \) is defined as the inf of such \( k \) (see e.g. [17], [18] and the references therein). The Young’s complement \( \tilde{\psi} \) of \( \psi \) is defined for \( u \in \mathbb{R} \) by \( \tilde{\psi}(u) := \sup_{v \geq u} \{|u| + \psi(v)\} \).

**Proposition 1.1.** (cf. [19]) Let \( \alpha \in (0, 1] \). For any Young’s function \( \psi \) with Young’s complement \( \tilde{\psi} \) satisfying

\[ \int_0^t \tilde{\psi}(s^{1-\alpha}) \, ds < \infty, \quad t \in [1, e], \quad (4) \]

the function \( \tilde{\Psi}: [0, \infty) \to [0, \infty) \) defined by

\[ \tilde{\Psi}(t) := \inf \{k > 0 : \int_0^t \tilde{\psi}(s^{\alpha-1}) \, ds \leq k t^{1-\alpha} \}, \quad t \geq 0, \quad (5) \]

is increasing and continuous with \( \tilde{\Psi}(0) = 0 \).

**Remark.** We remark that: If we fix \( p > 1 \), it can be easily seen that (4) holds with \( \tilde{\psi}_p(u) = \frac{|u|^p}{p} \) for all \( \alpha \in (\frac{1}{p}, 1) \). And if fix \( \alpha \in (0, 1) \), then (4) holds with \( \tilde{\psi}_p(u) = \frac{|u|^p}{p} \) for all \( p > \frac{1}{\alpha} \). However, if \( p > \frac{1}{\alpha} \), \( \alpha \in (0, 1) \), it is not hard to see that

\[ \tilde{\Psi}_p(t) = \frac{t^{\alpha-\frac{1}{p}}}{\sqrt[\alpha]{\tilde{\psi}(\alpha - 1) + 1}}, \quad t \geq 0 \quad (6) \]
This seems to be a good place to put the following observation.

**Proposition 1.2.** The Hadamard-type fractional integral operator maps the a.e. nonnegative, nondecreasing functions into functions of the same type.

**Proof.** Let \( t_1, t_2 \in [1, e] \) and \( x \) be a.e. nonnegative, nondecreasing function. With no loss of generality, we may assume that \( t_1 < t_2 \). Now, with some further efforts one can get

\[
\begin{align*}
\mathcal{J}^\alpha x(t_1) &= \frac{1}{\Gamma(\alpha)} \int_1^{t_1} (\log \frac{t_1}{s})^{n-1} x(s) \frac{ds}{s} = \frac{1}{\Gamma(\alpha)} \int_1^{t_1} (\log s)^{n-1} x\left(\frac{t_1}{s}\right) \frac{ds}{s} \\
&\leq \frac{1}{\Gamma(\alpha)} \int_1^{t_2} (\log s)^{n-1} x\left(\frac{t_2}{s}\right) \frac{ds}{s} = \mathcal{J}^\alpha x(t_2).
\end{align*}
\]

This yields \( 0 \leq \mathcal{J}^\alpha x(t_1) \leq \mathcal{J}^\alpha x(t_2) \) for any a.e. nonnegative, nondecreasing function \( x \), which is what we wished to show. \( \square \)

Now, we recollect the construction of the measure of noncompactness which will be used in the next section (see [20], [21]).

Let us fix a nonempty and bounded subset \( X \) of \( C[1, e] \). For \( x \in X \) and \( \epsilon \geq 0 \) denoted by \( \omega(x, \epsilon) \), the modulus of continuity of the function \( x \), i.e.,

\[
\omega(x, \epsilon) = \sup \{|x(t) - x(s)| : t, s \in [1, e], |t - s| \leq \epsilon\}.
\]

Further, let us put

\[
\omega(X, \epsilon) = \sup \{\omega(x, \epsilon) : x \in X\}, \quad \omega_0(X) = \lim_{\epsilon \to 0} \omega(X, \epsilon).
\]

Define

\[
\beta(x) := \sup \{|x(s) - x(t)| - |x(s) - x(t)| : t, s \in [1, e], t \leq s\},
\]

\[
\beta(X) := \sup \{\beta(x) : x \in X\}.
\]

Let us define the function \( \Lambda \) on the family of all nonempty and bounded subsets of \( C[1, e] \) by the formula \( \Lambda(X) := \omega_0(X) + \beta(X) \). The function \( \Lambda \) is a measure of noncompactness in the space \( C[1, e] \) [2].

**Remark.1** All functions belonging to \( X \) are nondecreasing on \([1, e]\) if and only if \( \beta(X) = 0 \). Now, let us conclude the introduction by stating main theorems that will be used in the sequel ([21], [22], [10], [23] and [24]).

**Theorem 1.1.** *(Darbo Fixed Point Theorem)* Let \( Q \) be a nonempty, bounded, closed and convex subset of the space \( E \) and let \( T : Q \to Q \) be \( \Lambda \)-condensing (i.e., it is a continuous and there exists a constant \( 0 \leq \lambda < 1 \) such that \( \lambda \Lambda(TX) \leq \Lambda(X) \) for any nonempty subset \( X \) of \( Q \)). Then \( T \) has a fixed point in the set \( Q \).

**Theorem 1.2.** *(Rothe Fixed Point Theorem)* Let \( Q \) be an open, bounded and convex subset of a Banach space \( X \) and let \( T : Q \to X \) be \( \Lambda \)-condensing. Then \( T \) has a fixed point if the following condition holds: \( 0 \in Q \) and \( T(\partial Q) \subseteq Q \).

## 2 Existence of Monotonic Continuous Solutions

In this section, we prove the existence of monotonic continuous solutions for equation (1) in \( C[1, e] \).

To facilitate our discussion, let us first state the following assumptions

1. \( h : [1, e] \to \mathbb{R}^+ \) is continuous, a.e. non-decreasing.
2. \( a : [1, e] \rightarrow \mathbb{R}^+ \) is measurable, a.e. non-decreasing form \( L_\psi([1, e]) \), where \( \psi \) is Young-type function satisfies the condition of Proposition 1.1.

3. \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^+ \) is continuous and non-decreasing with respect to the ordering in \( \mathbb{R}^2 \),

4. The operator \( G : C[1, e] \rightarrow C[1, e] \) is positive, continuous and satisfies the conditions of Theorem 1.1 for the measure of noncompactness \( \Lambda \),

5. There exist nonnegative constants \( c \) and \( d \) such that \(|(Gx)(t)| \leq c + d \|x\|\), holds for all \( x \in C[1, e] \),

6. Let \( \alpha > 0 \). Assume that the inequality \( \|h\| + 2(c + d \|x\|_\psi) f(r, (e - 1)r) \leq r \) has a positive solution \( r_0 \) such that \( \frac{2\psi(1)}{1}\|a\|_\psi f(r, (e - 1)r) < 1 \).

**Remark 2** Assumptions (2) and (3) imposed on \( f \) and \( a \) gives a guarantee that the map \( t \rightarrow a(t)f(\max_{s \in [1, t]} |x(s)|, \int_1^t |x(s)| ds) \) is non-decreasing on \([1, e]\) for any \( x \in C[1, e] \). Moreover,

\[
a(t)f(\max_{s \in [1, t]} |x(s)|, \int_1^t |x(s)| ds) \leq a(t)f(\|x\|, (e - 1)\|x\|), \text{ a.e. } t \in [1, e].
\]

Now, we are in a position to formulate and prove the following existence result.

**Theorem 2.1.** Let \( \psi \) be Young’s function with Young’s complement \( \tilde{\psi} \) satisfies (4). If the assumptions (1) – (6) holds, then quadratic integral equation (1) has at least one a.e. nondecreasing solution \( x \in C[1, e] \).

**Proof.** Let \( r_0 \) be any positive number satisfies Assumption (6). Define the nonempty, bounded, closed and convex set \( Q \) (required by Darbo’s fixed theorem) by:

\[
Q := \{ x \in C[1, e] : \|x\| \leq r_0 \}.
\]

Also, we define the operators \( F \) and \( T : Q \rightarrow Q \) by

\[
(Fx)(t) = f(H_1x(t), H_2x(t)),
\]

\[
Tx(t) = h(t) + Gx(t)\tilde{\psi}(a(t)(Fx)(t)), \quad t \in [1, e], \quad \alpha > 0,
\]

where \( H_1x(t) := \max_{s \in [1, t]} |x(s)| \) and \( H_2x(t) = \int_1^t |x(s)| ds \).

Observe that in view of our assumptions, for any function \( x \in C[1, e] \) the function \( Fx \) is continuous on \( \mathbb{R}^2 \).

Clearly, our assumptions imposed on \( a, h, Fx \) and \( G \) along with [25, Lemma 2.32.] give a reason to believe that the operator \( T \) makes sense.

We need to divide the proof into a few steps. In fact, we will prove the following four claims:

1. \( T : Q \rightarrow Q \) is well-defined,

2. \( T : Q \rightarrow Q \) is continuous operator,

3. \( T : Q \rightarrow Q \) is \( \Lambda \) – condensed.
To prove the assertion of (1), let \( x \in C[1,e] \) and \( t_1, t_2 \in [1,e] \) with \( t_1 \leq t_2 \). In the view of our assumptions we conclude that

\[
|Tx(t_2) - Tx(t_1)| \leq |h(t_2) - h(t_1)| + |Gx(t_2)|^\alpha a(t_2)(Fx)(t_2) - Gx(t_1)|^\alpha a(t_1)(Fx)(t_1)|
\]

\[
\leq |h(t_2) - h(t_1)| + |Gx(t_2)|^\alpha a(t_2)(Fx)(t_2) - Gx(t_1)|^\alpha a(t_1)(Fx)(t_1)|
\]

\[
\leq |h(t_2) - h(t_1)| + \frac{|Gx(t_2)|}{\Gamma(\alpha)} \int_1^{t_1} \left| (\log \frac{t_2}{s})^\alpha - (\log \frac{t_1}{s})^\alpha \right| a(s)(Fx)(s) \frac{ds}{s}
\]

\[
= |h(t_2) - h(t_1)| + \frac{|Gx(t_2)|}{\Gamma(\alpha)} \int_1^{t_1} \left| (\log \frac{t_2}{s})^\alpha - (\log \frac{t_1}{s})^\alpha \right| a(s)(Fx)(s) \frac{ds}{s}
\]

\[
\leq |h(t_2) - h(t_1)| + \frac{|Gx(t_2)|}{\Gamma(\alpha)} \int_1^{t_1} \left| (\log \frac{t_2}{s})^\alpha - (\log \frac{t_1}{s})^\alpha \right| a(s)(Fx)(s) \frac{ds}{s}
\]

\[
\leq |h(t_2) - h(t_1)| + \frac{|Gx(t_2)|}{\Gamma(\alpha)} \int_1^{t_1} \left| (\log \frac{t_2}{s})^\alpha - (\log \frac{t_1}{s})^\alpha \right| a(s)(Fx)(s) \frac{ds}{s}
\]

\[
\leq |h(t_2) - h(t_1)| + \frac{|Gx(t_2)|}{\Gamma(\alpha)} \int_1^{t_1} \left| (\log \frac{t_2}{s})^\alpha - (\log \frac{t_1}{s})^\alpha \right| a(s)(Fx)(s) \frac{ds}{s}
\]

\[
\leq |h(t_2) - h(t_1)| + \frac{|Gx(t_2)|}{\Gamma(\alpha)} \int_1^{t_1} \left| (\log \frac{t_2}{s})^\alpha - (\log \frac{t_1}{s})^\alpha \right| a(s)(Fx)(s) \frac{ds}{s}
\]

\[
\leq |h(t_2) - h(t_1)| + \frac{|Gx(t_2)|}{\Gamma(\alpha)} \int_1^{t_1} \left| (\log \frac{t_2}{s})^\alpha - (\log \frac{t_1}{s})^\alpha \right| a(s)(Fx)(s) \frac{ds}{s}
\]

\[
\leq |h(t_2) - h(t_1)| + \frac{|Gx(t_2)|}{\Gamma(\alpha)} \int_1^{t_1} \left| (\log \frac{t_2}{s})^\alpha - (\log \frac{t_1}{s})^\alpha \right| a(s)(Fx)(s) \frac{ds}{s}
\]

where

\[
M_1(s) := \begin{cases} \frac{(\log \frac{t_2}{s})^\alpha - (\log \frac{t_1}{s})^\alpha}{s} & s \in [1,t_2], \\ 0 & \text{otherwise} \end{cases} \quad M_2(s) := \begin{cases} \frac{(\log \frac{t_2}{s})^\alpha - (\log \frac{t_1}{s})^\alpha}{s} & s \in [1,t_2], \\ 0 & \text{otherwise} \end{cases}
\]

and

\[
M_3(s) := \begin{cases} \frac{(\log \frac{t_2}{s})^\alpha - (\log \frac{t_1}{s})^\alpha}{s} & s \in [1,t_1], \\ 0 & \text{otherwise} \end{cases}
\]

We claim that \( M_i \in L_\psi ^\alpha, \ i = 1, 2, 3 \). Once our claim is established, we conclude (in view of Hölder inequality in Orlicz space) that

\[
|Tx(t_2) - Tx(t_1)| \leq |h(t_2) - h(t_1)| + f(r, (e - 1)r) \left( \frac{2 \|a\|_\psi \|M_1\|_\psi}{\Gamma(\alpha)} \cdot |Gx(t_2) - Gx(t_1)| \right)
\]

\[
+ \frac{e + d \|x\|}{\Gamma(\alpha)} \cdot 2\|M_2\|_\psi + \|M_3\|_\psi \|a\|_\psi \right)
\]

\[
(8)
\]

\[
|Tx(t_2) - Tx(t_1)| \leq |h(t_2) - h(t_1)| + f(r, (e - 1)r) \left( \frac{2 \|a\|_\psi \|M_1\|_\psi}{\Gamma(\alpha)} \cdot |Gx(t_2) - Gx(t_1)| \right)
\]

It remain to prove our claim by showing that \( M_i \in L_\psi ^\alpha, \ i = 1, 2, 3 \). To see this fix \( 1 \leq t_i \leq t_2 \leq e \) and \( k > 0 \). An appropriate substitution using the properties of Young’s functions leads to the
following estimate
\[
J_1 = \int_1^t \frac{(\log \frac{t}{s})^{\alpha - 1}}{k s} \, ds \\
\leq \left( \frac{1}{k} \right)^{\frac{1}{\alpha}} \int_0^k \frac{1}{\alpha} \log(t_2) \psi(s^{\alpha - 1}) \, ds.
\]
then (in account of (5) and the definition of norm in Orlicz space) we have,
\[
\|M_1\|_{\psi} = \inf\{ k > 0 : J_1 \leq 1 \} \leq \Psi(\log t_2).
\]
Arguing similarly, we arrive at
\[
\|M_2\|_{\psi} \leq \Psi(|\log t_2 - \log t_1|) \leq \Psi(|t_2 - t_1|).
\]
Also, with some further efforts one can get
\[
J_3 = \int_1^t \psi \left( \frac{(\log \frac{t}{s})^{\alpha - 1} - (\log \frac{t}{1})^{\alpha - 1}}{k s} \right) \, ds \\
\leq \int_1^t \psi \left( \frac{(\log \frac{t}{s})^{\alpha - 1}}{k} \right) \, ds - \int_1^t \psi \left( \frac{(\log \frac{t}{1})^{\alpha - 1}}{k} \right) \, ds \\
= k^{\frac{1}{\alpha}} \int_0^{\frac{1}{k} \log(t_1)} \psi(s^{\alpha - 1}) \, ds - k^{\frac{1}{\alpha}} \log(\frac{1}{k}) \int_0^{\frac{1}{k} \log(t_2)} \psi(s^{\alpha - 1}) \, ds \\
\leq \left( \frac{1}{k} \right)^{\frac{1}{\alpha}} \int_0^{\frac{1}{k} \log(t_2)} \psi(s^{\alpha - 1}) \, ds.
\]
Hence
\[
\|M_3\|_{\psi} = \inf\{ k > 0 : J_3 \leq 1 \} \leq \Psi(|\log t_2 - \log t_1|) \leq \Psi(|t_2 - t_1|).
\]
Substituting into (8), one has,
\[
|Tx(t_2) - Tx(t_1)| \leq |h(t_2) - h(t_1)| \\
+ f(r, (e - 1)r) \left( \frac{2 \|a\|_{\psi} \Psi(\log t_2)}{1 \alpha} \cdot |Gx(t_2) - Gx(t_1)| \right) \\
+ \frac{e + d \|x\|}{1 \alpha} \cdot 4 \|a\|_{\psi} \Psi(|t_2 - t_1|).
\]
This may be combined with Proposition 1.1 in order to assure that
\[
|Tx(t_2) - Tx(t_1)| \to 0 \text{ as } t_2 \to t_1.
\]
That is, for every \(x \in Q\) for the \(Tx \in C[i, 1]\). Moreover, for any let \(x \in Q\) we have
\[
\|Tx\| \leq \|h\| + \frac{(e + d \|x\|)}{1 \alpha} f(r, (e - 1)r) \int_1^t \|((\log \frac{t}{s})^{\alpha - 1})|a(s)| \, ds \\
\leq \|h\| + \frac{2(e + dr)}{1 \alpha} f(r, (e - 1)r) \Psi(\log t) \|a\|_{\psi} \\
\leq \|h\| + \frac{2(e + dr)}{1 \alpha} f(r, (e - 1)r) \Psi(1) \|a\|_{\psi}.
\]
From which it follows, in view of Assumption (6), that \(T\) maps \(Q\) into itself. This establishes the first claim.
In this connection, it can be easily seen that $\epsilon > X$ for the assertion of (2).

According to Proposition T, for the assertion of (2), it is sufficient to choose $x_n \to x$ in $C[1, \epsilon]$. In this case a direct calculations yields

\[
|Tx_n(t) - Tx(t)| \leq \frac{\|Gx_n - Gx\|}{\Gamma(\alpha)} \int_1^\epsilon (\log \frac{t}{s})^{\alpha-1} a(s)(Fx_n)(s) \, ds + \frac{\|Gx\|}{\Gamma(\alpha)} \int_1^\epsilon (\log \frac{t}{s})^{\alpha-1} a(s)(Fx_n)(s) - (Fx)(s) \, ds
\]

\[
\leq \frac{\|Gx_n - Gx\|}{\Gamma(\alpha)} 2\|a\|_\psi \Psi(1)f(r, (e-1)r)
\]

\[
+ \frac{(c+dr)}{\Gamma(\alpha)}^2 \|a\|_\psi \sup_{t_2} \|(Fx_n)(t) - (Fx)(t)\|
\]

Hence, in view of our assumptions, we conclude that $T : Q \to Q$ is continuous operator as needed for the assertion of (2).

Finally, to prove the assertion of (3) let us take a nonempty set $X \subset Q$. Fix arbitrarily a number $\epsilon > 0$ and choose $x \in X$ and $t_1, t_2 \in [1, \epsilon]$ such that $t_1 \leq t_2$, $|t_1 - t_2| \leq \epsilon$. In view of equation (9) along with our assumptions, it follows

\[
\omega_0(Tx) \leq 2\Psi(1) \frac{1}{\Gamma(\alpha)} \|a\|_\psi f(r, (e-1)r)\omega_0(Gx).
\]

In this connection, it can be easily seen that

\[
|Tx(t_2) - Tx(t_1)| - |Tx(t_2) - Tx(t_1)| \leq \{|h(t_2) - h(t_1)| - |h(t_2) - h(t_1)|\}
\]

\[
+ |Gx(t_2)\beta^a(t_2)(Fx)(t_2) - Gx(t_1)\beta^a(t_1)(Fx)(t_1)|
\]

\[
- |Gx(t_2)\beta^a(t_2)(Fx)(t_2) - Gx(t_1)\beta^a(t_1)(Fx)(t_1)|
\]

\[
+ |Gx(t_1)\beta^a(t_2)(Fx)(t_2) - Gx(t_1)\beta^a(t_1)(Fx)(t_1)|
\]

\[
- |Gx(t_1)\beta^a(t_2)(Fx)(t_2) - Gx(t_1)\beta^a(t_1)(Fx)(t_1)|
\]

\[
+ |Gx(t_1)\beta^a(t_2)(Fx)(t_2) - Gx(t_1)\beta^a(t_1)(Fx)(t_1)|
\]

\[
\leq \{|Gx(t_2) - Gx(t_1)| - |Gx(t_2) - Gx(t_1)|\} 2\Psi(1) \frac{1}{\Gamma(\alpha)} \|a\|_\psi f(r, (e-1)r)
\]

\[
+ Gx(t_1)\beta^a(t_2)(Fx)(t_2) - Gx(t_1)\beta^a(t_1)(Fx)(t_1)|
\]

\[
- |\beta^a(t_2)(Fx)(t_2) - Gx(t_1)\beta^a(t_1)(Fx)(t_1)|
\]

According to Proposition 1.2, by taking into account Remark 2, we conclude that

\[
|Tx(t_2) - Tx(t_1)| - |Tx(t_2) - Tx(t_1)| \leq \{|Gx(t_2) - Gx(t_1)| - |Gx(t_2) - Gx(t_1)|\} 2\Psi(1) \frac{1}{\Gamma(\alpha)} \|a\|_\psi f(r, (e-1)r)
\]

\[
= \frac{2\Psi(1)}{\Gamma(\alpha)} \|a\|_\psi f(r, (e-1)r)\beta(Gx).
\]

The above inequality shows that

\[
\beta(Tx) \leq \frac{2\Psi(1)}{\Gamma(\alpha)} \|a\|_\psi f(r, (e-1)r)\beta(Gx).
\]
Consequently,
\[
\beta(TX) \leq \frac{2\hat{\psi}(1)}{\Gamma(\alpha)} \|a\| \cdot f(r, (e - 1)r) \beta(GX). \tag{12}
\]
Moreover, from inequalities (11) and (12) and the definition of the measure of noncompactness \(\Lambda\), we obtain
\[
\Lambda(TX) \leq \frac{2\hat{\psi}(1)}{\Gamma(\alpha)} \|a\| \cdot f(r, (e - 1)r) \Lambda(GX) \leq \frac{2\Lambda\hat{\psi}(1)}{\Gamma(\alpha)} \|a\| \cdot f(r, (e - 1)r) \Lambda(X). \tag{13}
\]

We have therefore shown that \(T : Q \rightarrow Q\) is a closed operator, hence by Darbo’s fixed point theorem \(T : Q \rightarrow Q\) has a fixed point \(x \in \mathbb{Q}\).

Now, let \(\text{Fix}(X)\) denotes the set of solutions of the integral equation (1). In the view of equation (13) we have \(\Lambda(\text{Fix}(X)) = 0\) which implies that \(\beta(\text{Fix}(X)) = 0\). Then (cf. Remark 1) all functions belonging to \(\text{Fix}(X)\) are nondecreasing, which is what we wished to show.

A similar existence result follows by applying Rothe’s fixed point theorem as follows.

**Theorem 2.2.** In Theorem 2.1 replace assumption (6) by the following:

(6’) Let \(\alpha > 0\). Assume that the inequality \(\|h\| + 2\frac{(c + dr)}{\Gamma(\alpha)} \hat{\psi}(1) \|a\| \cdot f(r, (e - 1)r) \geq r\) has a positive solution \(r_0\) such that
\[
\max \left( \frac{d\hat{\psi}(1)}{\Gamma(\alpha)} \|a\| \cdot f(r_0, (e - 1)r_0), \frac{\hat{\psi}(1)}{\Gamma(\alpha)} \|a\| \cdot f(r_0, (e - 1)r_0) \right) < \frac{1}{2}.
\]
Then equation (1) has at least one nondecreasing solution \(x \in C(0, 1]\).

**Proof.** First, note that Assumption (6’) suggests
\[
r_0 \leq \frac{\|h\| + 2c(\hat{\psi}(1) \|a\| \cdot f(r, (e - 1)r))}{1 - 2d(\hat{\psi}(1) \|a\| \cdot f(r, (e - 1)r))}. \tag{14}
\]
Let \(Q := \{x \in C[1, e] : \|x\| < r_0\}\) and \(x \in \partial Q\) i.e. \(\|x\| = r_0\). Then from inequality (10) we have
\[
\|Tx\| \leq \|h\| + 2\frac{(c + dr)}{\Gamma(\alpha)} \hat{\psi}(1) \|a\| \cdot f(r_0, (e - 1)r_0).
\]
This may be combined with inequality (14) in order to assure that \(\|Tx\| \leq r_0\). That is, \(T(\partial Q) \subset Q\).

Now, we are able to repeat the rest of the proof of the Theorem 2.1 and to apply Rothe fixed point theorem. Hence the claim follows.

We close our paper by introducing the following examples, which illustrate the results proved in Theorem 2.1 and does not apply the conditions in (cf. [4], [12] and [13]). We start with the following

**Example 1**
\[
x(t) = \frac{t}{20} + \frac{x(t)}{12} + \int_1^t (\log \frac{t}{s})^{-0.5} \log \left( \frac{e - 1}{e - s} \right) \left( \frac{e}{10} + \max_{u \in [1, s]} |x(u)| + \int_1^s x(u) du \right) \frac{ds}{s}, \quad t \in [1, e],
\]
Observe that the equation (15) is a special case of equation (1) if we put \(\alpha = 1/2\), \(h(t) = \frac{t}{20}\), \(a(t) = \log \left( \frac{e - 1}{e - t} \right)\) and
\[
f(H_1x(t), H_2x(t)) = \log \left( \frac{e}{10} + \max_{x \in [1, t]} |x(s)| + \int_1^t x(s) ds \right).
\]
and the operator $G$

Observe that the equation $(\|a\|_\psi = 2)$. Moreover, a direct calculation leads to

\[
\int_0^2 \psi(s^{-0.5}) ds \leq 3^2.
\]

So, owing to the definition of $\tilde{\Psi}$, we conclude that $\tilde{\Psi}(1) \leq 3$.

Now we observe that, the inequality

\[
\|h\| + 2 \left( \frac{e + dr}{\Gamma(\alpha)} \right) \tilde{\Psi}(1) \|a\|_\psi f(r, (e - 1)r) \leq r
\]

has the expression

\[
\frac{e}{20} + r(1 + \log(0.1 + r)) \leq r
\]

or equivalently

\[
\frac{e}{20} + r \log(0.1 + r) \leq 0
\]

It is easy to check that, the number $\frac{e}{20} < r_0 \leq (e^{-0.5} - 0.1)$ is positive solution of the above inequality. Furthermore, satisfies the Darbo condition with $\lambda = (\Gamma(\frac{1}{2})/6)$ then

\[
2\lambda \tilde{\Psi}(1) \|a\|_\psi^2 f(r_0, (e - 1)r_0) \approx (2(1 + \log(0.1 + r_0))) < 1.
\]

**Example 2**

\[
x(t) = \frac{t}{e^2} + \frac{1}{8e\sqrt{e} - 1} (1 + x(t)) \int_1^t \left( \frac{\log \left( \frac{1}{2} \right)}{\sqrt{s} - 1} \right) \left[ \max_{s \in |1,t]} |x(u)| + \int_1^s x(u) du \right] \frac{ds}{s}, \quad t \in [1, e], \quad (16)
\]

Observe that the equation (16) is a special case of equation (1) if we put $\alpha = \frac{3}{2}, h(t) = \frac{t}{e^2}$ and $a(t) = \frac{\sqrt{e} - 1}{8e\sqrt{e} - 1}$

\[
f(H_1 x(t), H_2 x(t)) = \max_{s \in |1,t]} |x(s)| + \int_1^t x(s) ds \quad \text{hence } f(r, (e - 1)r) = e \cdot r
\]

and the operator $G$ is given by $Gx(t) = \frac{r(\frac{1}{2})}{6e\sqrt{e} - 1} (1 + x(t))$

Thus, we note that $c = d = \frac{r(\frac{1}{2})}{6e\sqrt{e} - 1}$ and $a \in L_\psi[1, e], r > 0$, where $\psi(u) = \frac{|u|^2}{2}$

\[
\int_1^e \psi(u) \left( \frac{a(t)}{k} \right) dt = \frac{\sqrt{e} - 1}{k^2} \Rightarrow \|a\|_{\psi_2} = \sqrt{e} - 1.
\]

Hence, owing to (6), we have $\tilde{\Psi}_2(1) = 1$, and the inequality

\[
\|h\| + 2 \left( \frac{e + dr}{\Gamma(\alpha)} \right) \tilde{\Psi}_2(1) \|a\|_{\psi_2} f(r, (e - 1)r) \leq r
\]
has the expression
\[ \frac{1}{e} + \frac{r(1 + r)}{4} \leq r \]
or equivalently
\[ \frac{1}{e} + \frac{r^2}{4} - \frac{3r}{4} \leq 0 \]
It is easy to check that, the number 
\[ \frac{3/4 - \sqrt{9/16 - 1/e}}{1/2} \leq r_0 \leq \frac{3/4 + \sqrt{9/16 - 1/e}}{1/2} \]
is positive solution of the above inequality.

Competing Interests
Authors have declared that no competing interests exist.

References


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