Conformal Vector Fields on Finsler Space with Special $(\alpha, \beta)$-Metric

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Authors’ contributions
This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract
In this paper, we study the conformal vector fields on a class of Finsler metrics. In particular, Finsler space with special $(\alpha, \beta)$-metric $F = \alpha + \frac{\beta^2}{\alpha}$ is defined in Riemannian metric $\alpha$ and 1-form $\beta$ and its norm. Then we characterize the PDE’s of conformal vector fields on Finsler space with special $(\alpha, \beta)$-metric.

Keywords: Finsler space; conformal vector fields; special $(\alpha, \beta)$-metric.

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1 Introduction
The conformal theory of curves on Finsler geometry, emphasizing on the notion of circles preserving transformations, recently studied by the authors Z. Shen and Xia have studied conformal vector...
fields on a Randers manifold with certain curvature properties. Next, the conformal changes of metrics which leave invariant geodesic circles known as concircular transformation are characterized by a second order differential equation. The conformal vector fields are important in Riemann-Finsler geometry. Let \((M,F)\) be a Finsler manifold. It is known that a vector field \(v = v^i \frac{\partial}{\partial x^i}\) on \(M\) is a conformal vector field on \(F\) with conformal factor \(c = c(x)\) if and only if \(X_v(F^2) = 4cF^2\), where \(X_v = v^i \frac{\partial}{\partial x^i} + y^i \frac{\partial v}{\partial x^i} \partial_{y^j}\) [1]. They also determine conformal vector fields on a locally projectively flat Randers manifold. Besides they use homothetic vector fields \((c = \text{constant})\) on Randers manifolds to construct new Randers metrics of scalar flag curvature [2]. Randers metrics seem to be among the simplest non-trivial Finsler metrics with many investigation in Physics, Electron optics with a magnetic field, dissipative mechanics, irreversible thermodynamics etc.

In this paper, we shall study the conformal vector fields on Finsler space with special \((\alpha, \beta)\)-metric, whose metric is defined in Riemannian metric \(\alpha\) and 1-form \(\beta\) and its norm. Then we characterize the PDE’s of conformal vector fields on Finsler space with special \((\alpha, \beta)\)-metric. In natural way, we consider the general \((\alpha, \beta)\)-metrics are defined as the form:

\[
F = \alpha \phi(b^2, \frac{\beta}{\alpha}).
\]  

This kind of metrics is first discussed by Yu and Zhu [3]. Many well-known Finsler metrics are general \((\alpha, \beta)\)-metrics. For example, the Randers metrics and the square metrics are defined by functions \(\phi = \phi(b^2, s)\) in the following form:

\[
\phi = \frac{\sqrt{1 - b^2 + s^2} + s}{1 - b^2},
\]

\[
\phi = \left(\frac{\sqrt{1 - b^2 + s^2} + s}{1 - b^2}\right)^{2\sqrt{1 - b^2 + s^2}}.
\]

Based on the some reviews, further we shall study the covariant derivatives of conformal vector field is directly proportional to Finsler Special \((\alpha, \beta)\)-metric.

2 Preliminaries

Let \(M\) be an \(n\)-dimensional differentiable manifold and \(TM\) be the tangent bundle. A Finsler metric on \(M\) is the function \(F = F(x, y) : TM \rightarrow R\) satisfying the following conditions:

1. \(F(x, y)\) is a \(C^\infty\) function on \(TM \setminus \{0\}\);
2. \(F(x, y) \geq 0\) and \(F(x, y) = 0 \rightarrow y = 0\);
3. \(F(x, \lambda y) = \lambda F(x, y), \lambda > 0\);
4. the fundamental tensor \(g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2(F^2)}{\partial y^i \partial y^j}\) is positively defined.

Let

\[
C_{ijk} = \frac{1}{2}[F^2]_{y^i y^j y^k} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i y^j y^k},
\]

Define symmetric trilinear form \(C = C_{ijk} dx^i \otimes dx^j \otimes dx^k\) on \(TM \setminus \{0\}\). We call \(C\) be the Cartan torison tensor.

Let \(F\) be a Finsler metric on an \(n\)-dimensional manifold \(M\). The canonical geodesic \(\sigma(t)\) of \(F\) is characterized by

\[
\frac{d^2 \sigma^i(t)}{dt^2} + 2G^i(\sigma(t), \dot{\sigma}(t)) = 0,
\]
Let $F$ be a Finsler metric on a manifold $M$, and $V$ be a vector field on $M$. Let
\[ G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}. \]

Given geodesic coefficients $G^i$, we define the covariant derivatives of a vector field as $XX^i(t) \frac{\partial}{\partial x^i}$ along a curve $c(t)$ by
\[
D_t X = \{X^i(t) + X^i(t)N^j(t)(c(t))\} \frac{\partial}{\partial x^i}(c(t)),
\]
where $N^i = \frac{\partial G^i}{\partial y^j}, X^i(t) = \frac{dx^i}{dt}$ and $\dot{c} = \frac{dx^i}{dt} \frac{\partial}{\partial x^i}$.

Let $F = \alpha + \frac{\sigma}{2}$ be a Finsler Special $(\alpha, \beta)$-metric expressed in terms of a Riemannian metric $\alpha$ and a vector field $V$ on $M$.

From equation (1.1), where $\phi = \phi(b^2, s)$ is a positive smooth function on $[0, b_0] \times (-b_0, b_0)$. It is required that
\[
\phi - \phi_2s > 0, \quad \phi - \phi_2s + (b^2 - s^2)\phi_{22} > 0, \quad (2.1)
\]
for $b < b_0$ where $\phi_1 = \frac{\partial \phi}{\partial s}, \phi_2 = \frac{\partial^2 \phi}{\partial s^2}, \alpha = \frac{\sqrt{1-b^2+s^2}}{1-b^2}, \beta = \frac{s}{1-b^2}$.

We write the function where $\phi = \phi(b^2, s)$ in the following Taylor expansion
\[
\phi = r_0 + r_1s + r_2s^2 + O(s^3),
\]
where
\[
r_1 = r_1(b^2), \quad r_0 = \frac{1}{(1-b^2)^2}, \quad r_1 = \frac{1}{1-b^2}, \quad r_2 = \frac{1}{2(1-b^2)^3}.
\]

Now (1.2) implies that
\[
r_0 > 0, \quad r_0 + 2b^2r_2 > 0.
\]
But there is no restriction on $r_1$. If we assume that $r_1 \neq 0$, then $F$ is not reversible. (since $F$ is not a symmetric see [4]).

Now, the Finsler Special $(\alpha, \beta)$ metric is on the conformal vector field, then Finsler Special $(\alpha, \beta)$-metric becomes
\[
\phi(b^2, s) = \frac{s^2}{(1-b^2)\sqrt{1-b^2+s^2}}
\]
and (1.2) and (1.3) satisfy
\[
\frac{1}{2b^2} \frac{r_{11}}{r_1} - \frac{r_{10}}{r_0} + \left\{ \frac{r_{22}}{r_0} \left[ \frac{r_{11}}{r_1} - \frac{r_{10}}{r_0} \right] - \frac{r_{12}}{r_0} \right\} b^2 = \frac{1}{2b^2(1-b^2)^3}.
\]

**Definition 2.1.** Let $F$ be a Finsler metric on a manifold $M$, and $V$ be a vector field on $M$. Let $\phi_t$ be the flow generated by $V$. Define $\phi_t : TM \to TM$ by $\phi_t(x, y) = (\phi_t(x), \phi_t(\gamma))$. Then the vector field $V$ is said to be conformal if
\[
\phi_t^* F = e^{-2\sigma_t} F, \quad (2.3)
\]
where $\sigma_t$ is a function on $M$ for every $t$. Differentiating the above equation by $t$ at $t = 0$, we obtain
\[
X_c(F) = -2cF, \quad (2.4)
\]
where $c$ is called the conformal factor and $X_c$ is covariant derivative of vector field $X$, it can be defined as
\[
X_c = V^i \frac{\partial}{\partial x^i} + y^i \frac{\partial V^j}{\partial x^i} \frac{\partial}{\partial y^j}, c = \frac{d}{dt} = 0\sigma_1. \quad (2.5)
\]
3 Conformal Vector Fields on Finsler Space with Special \((\alpha, \beta)\)-metric

In this section we shall study the conformal vector field on Kropina metric with (1.2). Let \(V\) be a conformal vector field of \(F\) with conformal factor \(c(x)\).

\[ i.e., X_c(F^2) = 4cF^2. \quad (3.1) \]

Now we are in the position from (1.2) and to solve the above with the Kropina metric, we have

\[ F = \alpha + \frac{\beta^2}{\alpha} = \frac{\alpha^2}{(1 - b^2)\sqrt{1 - b^2 + s^2}}, \]

then (3.1) implies

\[ X_c(F^2) = \phi^2 X_c(\alpha^2) + \alpha^2 X_c(\phi^2), \]

\[ X_c(F^2) = [A_1 X_c(\alpha^2) + 2\alpha^2 X_c(\beta^2)A_2 + 2\alpha X_c(\beta)A_3 - 2\beta X_c(\alpha)A_4], \quad (3.2) \]

where,

\[ A_1 = \frac{s^4}{(1 - b^2)\sqrt{1 - b^2 + s^2}}, \quad A_2 = \frac{s^4}{(1 - b^2)\sqrt{1 - b^2 + s^2}}, \]

\[ A_3 = 0, \quad A_4 = 0 \]

Then we have

\[ X_c(\alpha^2) = 2V_{0,0}, \quad X_c(\beta) = (V^j b_{i,j} + b^j V_{j,i})y^i. \]

Again equation (3.2) equivalent to

\[ B_1 V_{0,0} + \alpha B_2 (V^j b_{i,j} + b^j V_{j,i})y^i + s^2 \alpha^2 X_c(\beta^2) - 2B_3 c^2 = 0 \quad (3.3) \]

\[ B_1 = \frac{s^2}{(1 - b^2)\sqrt{1 - b^2 + s^2}}, \quad B_2 = 0, \]

\[ B_3 = \frac{s^2}{2} (1 - b^2)^2 (1 - b^2 + s^2)^{\frac{1}{2}} - \left\{ \frac{s^2}{(1 - b^2)\sqrt{1 - b^2 + s^2}} \right\}. \quad (3.4) \]

To simplify the computation, we fixed point \(x \in M\) and make a co-ordinate change such that

\[ y = \frac{s}{\sqrt{b^2 - s^2}} \xi, \quad \alpha = \frac{b}{b^2 - s^2} \xi, \quad \beta = \frac{bs}{b^2 - s^2} \xi, \quad \xi = \sum_{n=2}^n (y^n)^2. \]

Then we have

\[ V_{0,0} = V_{1,1} \frac{s^2}{b^2 - s^2} \xi + (V_{1,0} + V_{1,0,1}) \frac{s}{\sqrt{b^2 - s^2}} \xi + V_{0,0}, \quad (3.5) \]

\[ V^j b_i + b^j V_{i,j}y^i = (V^j b_{i,j} + b^j V_{j,i}) \frac{s}{\sqrt{b^2 - s^2}} \xi + (V^j b_{0,j} + b^j V_{j,0}), \quad (3.6) \]

where,

\[ \nabla_{1,0} + \nabla_{0,1} = \sum_{a=2}^n (V_{1,p} + V_{p,1})y^p, \quad \nabla_{0,0} = \sum_{p,q=0}^n V_{p,q}y^p y^q, \quad (3.7) \]
\[ V^j \bar{b}_{0,j} + b^j \nabla_{j,0} = \sum_{p=2}^{n} (V^j b_{p,j} + b^j V_{j,p}) y^p. \]

From (3.5) and (3.6) in to (3.3), which yields
\[
\begin{align*}
& B_1 \{ V_{1,1} \frac{s^2}{b^2 - s^2} \bar{\alpha}^2 + \left( V_{1,0} + V_{0,1} \right) \frac{s}{\sqrt{b^2 - s^2}} \bar{\pi} + V_{0,0} \} \\
& + B_2 \left\{ \frac{b}{\sqrt{b^2 - s^2}} \bar{\pi} \left\{ (V^j b_{1,j} + b^j V_{j,1}) \frac{s}{\sqrt{b^2 - s^2}} \bar{\pi} + (V^j b_{0,j} + b^j V_{j,0}) \right\} \\
& + b^j V_{j,0} \} + \{ s^2 X_v(b^2) - 2B_2 c \} \frac{b^2}{b^2 - s^2} \bar{\alpha}^2 = 0.
\end{align*}
\]

(3.8)

Consider the polynomial
\[
\phi = r_0 + r_1 s + r_2 s^2 + o(s^3)
\]
with \( r_i = r_i(b^2) \) then we have,
\[
\phi_1 = r_0^1 + r_1^1 s + r_2^1 s^2 + o(s^2).
\]

By letting \( s = 0 \) in (3.8) we get,
\[
V_0 \nabla_{0,0} + r_1 (V^j \bar{b}_{0,j} + b^j \nabla_{j,0}) \bar{\pi} + \{ r_0^1 X_v(b^2) - 2cr_0 \} \bar{\alpha}^2 = 0.
\]

(3.9)

According to the irrationality of \( \bar{\pi} \), the (3.8) is equivalent to
\[
\begin{align*}
& r_1 (V^j \bar{b}_{0,j} + b^j \nabla_{j,0}) = 0, \\
& r_0 (V_{0,0} + r_0^1 X_v(b^2) - 2cr_0) \bar{\alpha}^2 = 0.
\end{align*}
\]

(3.10)

Therefore, the equation (3.10) yields
\[
\begin{align*}
& (V^j \bar{b}_{0,j} + b^j \nabla_{j,0}) = 0, \\
& V^j b_{r,j} + b^j \nabla_{j,r} = 0.
\end{align*}
\]

(3.12)

Now, from equation (3.11), we have,
\[
V_{r,s} + V_{s,r} = -2\{ \frac{r_0^1}{r_0} X_v(b^2) - 2c \} \delta_{rs}, \quad 2 \leq r, s \leq n.
\]

(3.13)

Again irrationality of \( \bar{\pi} \) from (3.3) we get
\[
B_1 (V_{1,0} + V_{0,1}) \frac{s}{\sqrt{b^2 - s^2}} \bar{\pi} = 0.
\]

(3.14)

\[
B_1 \{ V_{1,1} \frac{b^2}{b^2 - s^2} \bar{\alpha}^2 + V_{0,0} \} + B_2 \left\{ \frac{bs}{b^2 - s^2} \bar{\alpha}^2 \left( V^j b_{1,j} + b^j V_{j,1} \right) \\
+ \{ s^2 X_v(b^2) - 2cB_2 \} \frac{b^2}{b^2 - s^2} \bar{\alpha}^2 \right\} = 0.
\]

(3.15)

From (3.13) we get
\[
V_{1,0} + V_{0,1} = 0.
\]
Which is equivalent to
\[ V_{0,x} + V_{r,1} = 0. \] (3.16)

Solving (3.11) for \( \nabla_{0,0} \) and plugging it in to (3.15) we have
\[
2B_1 s^2 \{V_{1,1} \frac{r_1^2}{r_0} (X_v(b^2) - 2c)\} - 2\{ \frac{r_1^2}{r_0} X_v(b^2) - 2c \} B_1(b^2) \\
+ B_2 s b (V' b_{1,j} + b^j V_{1,1}) + B_3 b^2 X_v(b^2) - 2c b^2 = 0. \] (3.17)

By Taylor series, expansion of \( \phi(b^2,s) \) then plugging it in to (3.15) and by the coefficients of \( s \) we have.
\[
br_1 (V'^2 b_{1,j} + b^j V_{1,1}) + b^2 X_v(b^2) \frac{\partial r_1}{\partial b} - 2c b^2 r_1 = 0. \] (3.18)

Then
\[
V'^2 b_{1,j} + b^j V_{1,1} = \left( \frac{r_1^2}{r_1^1} X_v(b^2) - 2c \right) b_i. \] (3.19)

From equation (3.13) and (3.19) which yields,
\[
V'^2 b_{1,j} + b^j V_{1,1} = \left( \frac{r_1^2}{r_1} X_v(b^2) - 2c \right) b_i. \] (3.20)

Substituting (3.19) in (3.18), we get,
\[
B_1 s^2 \{V_{1,j} + \left( \frac{p_1^1}{p_0^1} X_v(b^2) - 2c \right) - b^2 X_v(b^2) \left( \frac{p_1^1}{p_0^1} B_1 - B_2 \right) + B_3 s \frac{p_1^1}{p_1^1} \} = 0. \] (3.21)

The coefficients of all powers of \( s \) must vanish in (3.21). In particular, the coefficients of \( s^2 \) vanishes, the above equation becomes,
\[
V_{1,1} + \frac{p_1^1}{p_0^1} X_v(b^2) - 2c b = -b^2 X_v(b^2) R_0, \] (3.22)

where
\[
R_0 = \frac{p_1^1}{p_0^1} p_2 + \frac{p_1^1}{p_0^1} - 2 \frac{p_1^1}{p_1^1} \frac{p_1^1}{p_0^1}. \]

By (3.13),(3.16) and (3.22), we have
\[
V_{i,j} + V_{j,i} = 4c p_{i,j} - 2X_v(b^2) \left( \frac{p_1^1}{p_0^1} p_{i,j} + R_0 b_i b_j \right). \] (3.23)

It is equivalent to
\[
v_{i,j} + v_{j,i} = 4c a - 2X_v(b^2) \left( \frac{p_1^1}{p_0^1} a + R_0 \beta \right). \] (3.24)

On contracting (3.16) with \( b^i \) and \( b^j \) yields
\[
V_{i,j} b^i b^j = 2c b^2 - b^2 X_v(b^2) \left( \frac{p_1^1}{p_0^1} + R_0 \gamma \right). \] (3.25)

Which is equivalent to
\[
V_{i,j} b^i b^j = 2c b^2 - b^2 X_v(b^2). \]
Contracting (3.20) with $b^i$ and $b^j$ yields
\[ V_{ij}b^ib^j = 2eb^2 - b^2X_v(b^2)\{ 1 - \frac{p_1^2}{p_1} \}. \] (3.26)

Here, we used the fact that $X_v(b^2) = 2b_{ik}b^iV^k$. Then comparing (3.22) with (3.23), it yields
\[ X_v(b^2)\{ R_1 - R_0b^2 \} = 0, \] (3.27)

where $R_1 = \frac{1}{2b^2} + \frac{p_1^2}{p_1} - \frac{p_2}{p_2}$. Therefore (3.27) reduced to
\[ X_v(b^2)\{ R_1 + R_0b^2 \} = 0. \] (3.28)

Here, two cases arise:

Case 1: If
\[ R_1 + R_0b^2 \neq 0, \] (3.29)

where, $R_2 = \frac{p_1^2}{p_2} + \frac{p_1^2}{p_1} - 2\frac{p_1}{p_2}$.

It follows from (3.29) that $X_v(b^2) = 0$ and in (3.19) and we have
\[ V_{ij} + V_{ji} = 4e\alpha, \quad V^jb_{ij} + b^jV_{ji} = 2e\beta. \] (3.30)

Notice that if $X_v(b^2) = 0$ and (3.30) holds then $V$ satisfies (3.2) and $V$ is an conformal vector field. Therefore, we obtain

**Theorem 3.1.** Let $F = \frac{a^2}{b^2}$ be a Kropina metric on an $n$-dimensional manifold $M$ $(n \geq 3)$ and let $V = V^i(x)\frac{\partial}{\partial x^i}$ be a conformal vector field. Then $V$ is a conformal vector field of $F$ with conformal factor $c = c(x)$ iff $X_v(b^2) = 0$ and
\[ V_{ij} + V_{ji} = 4e\alpha, \quad V^jb_{ij} + b^jV_{ji} = 2e\beta. \] (3.31)

Case 2: If
\[ R_1 + R_0b^2 = 0. \] (3.32)

In this case $X_v(b^2) \neq 0$. Then obviously, we have
\[ V_{ij} + V_{ji} = 4e\alpha - 2X_v(b^2)\{ b^2R_1b^2 \}, \] (3.33)
\[ V^jb_{ij} + b^jV_{ji} = 2e\beta. \] (3.34)

Since $V$ is conformal vector field and from above equation (3.33) reduced to
\[ X_v(b^2)(B_1b^{-1}[b^2 - s^2]R_1^*) + B_2 - \left( 1 - \frac{b^2 + s^2}{s(1-b^2)} \right) \frac{p_1^2}{p_1} = 0. \] (3.35)

Then, we obtain the theorem;

**Theorem 3.2.** Therefore it follows we obtain Let $F = \frac{a^2}{b^2}$ be a Kropina metric on an $n$-dimensional manifold $M$ $(n \geq 3)$ and let $V = V^i(x)\frac{\partial}{\partial x^i}$ be a conformal vector field. Then, $V$ is a conformal vector field of $F$ with conformal factor $c = c(x)$ iff
\[ V_{ij} + V_{ji} = 4e\alpha - 2X_v(b^2)\{ b^2R_1b^2 \}, \] (3.36)
\[ V^jb_{ij} + b^jV_{ji} = 2e\beta, \] (3.37)
\[ X_v(b^2)(B_1b^{-1}[b^2 - s^2]R_1^*) + B_2 - \left( 1 - \frac{b^2 + s^2}{s(1-b^2)} \right) \frac{p_1^2}{p_1} = 0. \] (3.38)
where,

\[ R_1 = \left( \frac{1}{2b^2} + p_1^1 - \frac{p_0^3}{p_0} \right), \]

\[ R_1^* = \left( \frac{p_1^1}{p_1} - \frac{p_0^3}{p_0} \right) \frac{s^2}{2b^2}, \]

\[ B_1 = \frac{b^4(1 + s^2) - 4b^2}{s(1 - b^2)^2}, \]

\[ B_2 = \frac{\alpha^2 - b^4 - b^2(s^2 - 2) - 1}{s^2(1 - b^2)^2}, \]

\[ \tau = c - \frac{1}{2} X_v(b^2 \frac{p_0^3}{p_0}) \]

(3.39)

4 Conclusions

Conformal vector fields play an important role in Finsler geometry. When \( F \) is Riemannian metric, the local solutions of a conformal vector field can be determined if \( F \) satisfies certain conditions. As we know every conformal vector field is associated with scalar function called conformal factor.

In this paper we study the conformal vector field on Finsler - Kropina metric and characterize conformal vector fields on Kropina metric in terms of PDE’s.

Competing Interests

Authors have declared that no competing interests exist.

References