Wong-Zakai Method Applications for Explicitly Solvable Stochastic Differential Equations

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Authors’ contributions:

This work was carried out in collaboration between both authors. Author SS designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Authors SS and MM managed the analyses of the study. Author MM managed the literature searches. Both authors read and approved the final manuscript.

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Abstract

In this study, three Itô stochastic differential equations with multiplicative noise are investigated with Wong-Zakai method. The stochastic differential equations are also analyzed by Euler-Maruyama, Milstein and Runge Kutta stochastic approximation methods. The relative errors of these three methods are compared and the performance of Wong-Zakai method is shown alongside numerical results.

Keywords: Wong-Zakai method; stochastic differential equations; Euler method; Milstein method.

1 Introduction

Mathematical models in the literature are mostly investigated by the use of deterministic differential equations and systems of equations. However it is known that most real life events contain components that act non-deterministically, i.e. randomly. The use of random and stochastic differential equation systems to
model such events is a better modeling approach in contrast to the deterministic case. Several popular methods are used for analyzing stochastic differential equation systems, such as Euler-Maruyama and Milstein methods [1,2], which are the pioneering approximation techniques.

In this study, we present the Wong-Zakai approximation method [3] to several reducible stochastic differential equations (SDEs) that are explicitly solvable. The equations, which are given as Ito SDEs will be transformed to Stratonovich SDEs, since Wong-Zakai approximation method is based on Stratonovich stochastic integration technique. The most basic advantage of Wong-Zakai method is that the technique can be used together with deterministic numerical methods for approximation. Hence, a suitable approximation method can be selected for using with Wong-Zakai method depending on the type of problem that will be analyzed. Wong-Zakai method has been previously used for analyzing stochastic partial differential equations and SDEs driven by martingales [4,5]. Additionally, recent applications of Wong-Zakai approximation have been given stochastic Landau–Lifshitz–Gilbert and stochastic reaction-diffusion equations [6,7,8]. Stochastic differential equations have many applications in various research fields ranging from computer virus models in engineering [9] to Diabetes Mellitus models in medicine [10].

The study is structured as follows: In section 2, basic definitions and the methodology for Wong-Zakai method is given. Section 3 contains the problems for applications and section 4 contains the results for the examples. Lastly, conclusion is given along with the remarks of the study.

2 Methodology

In this section, Wong-Zakai approximation method is given along with the deterministic numerical approximation method, the predictor-corrector method for multi-step methods, and is compared with the well-known stochastic methods Euler-Maruyama, Milstein, strong 1.0 order Runge-Kutta method and stochastic Runge-Kutta method. These methods are presented below.

An Ito SDE can be given as

\[ dX_t = a(t,X_t)dt + b(t,X_t)dW_t \]  

(1)

on \([t_0, T]\) with an initial value \(X_{t_0} = x_0\). Euler-Maruyama, Milstein and strong 1.0 order Runge-Kutta methods are implemented to the Ito SDEs whereas the other methods are implemented to Stratonovich SDEs. The Stratonovich form of (1) can be given as

\[ dX_t = a(t,X_t)dt + b(t,X_t) \circ dW_t. \]

(2)

The relationship equations (1) and (2) is given as

\[ a(t,X_t) = a(t,X_t) - \frac{1}{2} b(t,X_t) \frac{\partial b}{\partial X_t}(t,X_t). \]

(3)

Let the interval \([t_0,T]\) be divided into \(k\) subintervals equally and the discretization points be denoted as \(t_0 < t_1 < \cdots < t_{k-1} < t_k = T\) to approximate the numerical solution of (1) or (2).

The Euler-Maruyama scheme [1,2] is given as

\[ \bar{X}_{t_{n+1}} = \bar{X}_{t_n} + a(t_n,\bar{X}_{t_n})\Delta t_n + b(t_n,\bar{X}_{t_n})\Delta W_n, \quad n = 0,1,2, \ldots, k-1 \]

(4)

with the initial value \(\bar{X}_{t_0} = X_0\). Here, \(\Delta t_n = t_{n+1} - t_n, \Delta W_n = W_{t_{n+1}} - W_{t_n}\). This method has strong order 0.5. The strong order 1.0 of Milstein scheme [1] is an improvement of Euler-Maruyama method and given as
\[ \bar{X}_{tn+1} = \bar{X}_{tn} + a(t_n, \bar{X}_{tn})\Delta t_n + b(t_n, \bar{X}_{tn})\Delta W_n + \frac{1}{2} b(t_n, \bar{X}_{tn})b(t_n, \bar{X}_{tn})(\Delta W_n)^2 - \Delta t_n \] (5)

where \( n = 0, 1, 2, ..., k - 1 \) and \( \bar{X}_{t_0} = X_0 \). Both Euler and Milstein schemes have weak order 1.0. The strong order 1.0 Runge-Kutta [1], is proposed by Platen, is given as

\[ \bar{X}_{tn+1} = \bar{X}_{tn} + a(t_n, \bar{X}_{tn})\Delta t_n + b(t_n, \bar{X}_{tn})\Delta W_n + \frac{1}{2\sqrt{\Delta t_n}} \left( b(t_n, \bar{Y}_{tn}) - b(t_n, \bar{X}_{tn}) \right) \left( (\Delta W_n)^2 - \Delta t_n \right) \] (6)

where \( \bar{Y}_{tn} = \bar{X}_{tn} + a(t_n, \bar{X}_{tn})\Delta t_n + b(t_n, \bar{X}_{tn})\Delta W_n \), \( n = 0, 1, ..., k - 1 \).

The second Runge-Kutta method [11], which is implemented to the Stratonovich SDE, is given similar to its deterministic counterpart but with an additional component for the stochastic noise approximation

\[ \bar{X}_{tn+1} = \bar{X}_{tn} + \frac{1}{6} \left( \left[ F_0 + 2F_1 + 2F_2 + F_3 \right] \Delta t_n \right. \]
\[ \left. + \left[ G_0 + 2G_1 + 2G_2 + G_3 \right] \Delta W_n \right) \] (7)

where

\[ F_0 = a(t_n, \bar{X}_{tn}), \]
\[ F_1 = a \left( t_n + \frac{1}{2} \Delta t_n, \bar{X}_{tn} + \frac{1}{2} F_0 \Delta t_n + \frac{1}{2} G_0 \Delta W_n \right), \]
\[ F_2 = a \left( t_n + \frac{1}{2} \Delta t_n, \bar{X}_{tn} + \frac{1}{2} F_1 \Delta t_n + \frac{1}{2} G_1 \Delta W_n \right), \]
\[ F_3 = a \left( t_n + \Delta t_n, \bar{X}_{tn} + F_2 \Delta t_n + G_2 \Delta W_n \right) \]

and \( G_i, i = 0,1,2,3 \) are similarly obtained by the evaluation of coefficient \( b(t, X_t) \) at the same point.

2.1 Wong-Zakai scheme

In this method, the SDE problem is reduced to an ODE problem at each interval \([t_j, t_{j+1}]\), \( j = 0, 1, ..., k - 1 \) and solved by an ODE solver. The ODE problem to obtain \( X_{t_{j+1}} \) for each interval \([t_j, t_{j+1}]\) is given by [12]

\[ \frac{d\hat{X}_t}{dt} = a(t, \hat{X}_t) + \frac{1}{\Delta t_n} b(t, \hat{X}_t)\Delta W_n \] (8)

with \( \hat{X}_{tn} = \hat{X}_j, j = 0, 1, ..., k - 1 \). Here, \( \Delta W_n = W_{t_{j+1}} - W_{t_j} \) is the discrete approximation of \( dW_t \). In this article, the predictor-corrector method, which is a kind of multistep method, is used to solve the ODE (8). Adams-Bashford method is used as the predictor and Adams-Moulton method is used as the corrector [13].

3 Numerical Examples

In this section, we implement the methods which are introduced in Section 2 to the three reducible SDE problems with known analytical solutions to compare the efficiency of the Wong-Zakai method to the other approximation methods. The expectation for the approximate solutions of the stochastic differential equations are found from the averaging of \( N \) simulated sample paths.
Here $X_{t_i}$ denotes the $i$-th sample path.

**Problem 1.** Given the Ito SDE as

$$dX_t = a^2X_t (1 + X_t^2)dt + a(1 + X_t^2)dW_t, \quad X_{t_0} = X_0, a \in \mathbb{R}.$$ 

The analytical solution of Problem 1 is [1]

$$X_t = \tan(aW_t + \arctan X_0).$$

The Stratonovich form of Problem 1 is obtained as

$$dX_t = a(1 + X_t^2) \circ dW_t.$$ 

**Problem 2.** Given the Ito SDE as

$$dX_t = \frac{1}{3}X_t^3 dt + X_t^2 dW_t, \quad X_{t_0} = X_0$$

the analytical solution and the Stratonovich form of the problem are given by

$$X_t = \left(\frac{1}{3}X_0^3 + \frac{1}{3}W_t\right)^3$$

$$dX_t = X_t^2 \circ dW_t$$

respectively [4].

**Problem 3.** For the Ito SDE

$$dX_t = -\sin(X_t) \cos^3(X_t) dt + \cos^2(X_t) dW_t, \quad X_{t_0} = X_0.$$ 

The Stratonovich form is

$$dX_t = \cos^2(X_t) \circ dW_t$$

and the analytical solution is [4]

$$X_t = \arctan(W_t + \tan(X_0)).$$

4 Results

In this section, we implement the numerical methods which were introduced in Section 2 to reducible SDE problems and compare the numerical efficiencies of the mentioned methods. For this, we calculate an error at point $t_i, i = 0,1,2,...,k$ as

$$e(t_i) = \frac{|X_{analytical} - X_{numerical}|}{|X_{analytical}|}$$
where \( X_{\text{analytical}} \) and \( X_{\text{numerical}} \) are the analytical and numerical solutions at the point of interest, respectively.

**Table 1. Comparison of results for the corresponding methods in problem 1**

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For all of the problems, we investigate the closed time interval \([0,3]\) and use \( N = 100,000 \) simulations for obtaining the expectations. Also, we take 61 points (of which 31 are shown in the tables) which equally divide this interval. Then, we obtain Table 1 for Problem 1 with the initial condition \( X_0 = 1 \).

All of the relative errors have been shown below for the methods in the figure (Fig. 1) along with the solution graphs of the corresponding methods (Fig. 2).

If the relative errors at \( t_f = 3.0 \) are investigated for problem 1, the following results are obtained:

\[
e_{EM} = 0.0010, \quad e_{\text{Milstein}} = 0.0001, \quad e_{RKIV} = 0.0001, \quad e_{RK1.0} = 0.0024
\]

respectively. The relative error for Wong-Zakai method is obtained as \( e_{WZ} = 0.0004 \), meaning that Wong-Zakai method out-performs the strong order 1.0 Runge-Kutta method and Euler-Maruyama method and provides similar results to Runge-Kutta IV and Milstein methods.
Fig. 1. Error plots for the methods in problem 1

Fig. 2. Solution plots for the methods in problem 1
For Problem 2, we have the numerical solutions which are shown the following Table 2 with the initial condition \( X_0 = 100 \).

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For Problem 3, the numerical results are shown in the following Table 3 with the initial condition \( X_0 = 0.5 \).

If the relative errors at \( t_1 = 3.0 \) are investigated for problem 2, the following results are obtained:

\[
e_{EM} = 0.0006, \quad e_{Milstein} = 0.0002, \quad e_{RKIV} = 0.0851 \times 10^{-3}, \quad e_{RK1.0} = 0.0018
\]

respectively. The relative error for Wong-Zakai method is obtained as \( e_{WZ} = 0.0001 \), meaning that Wong-Zakai method gives the numerical approximation with the least error except Runge-Kutta IV compared to the analytical solution for problem 2. It is seen that Runge-Kutta IV and Wong-Zakai give similar relative errors for this problem. All of the relative errors have been shown below for the methods in the figure (Fig. 3) along with the solution graphs of the corresponding methods (Fig. 4).
If the relative errors at \( t_i = 3.0 \) are investigated for problem 3, the following results are obtained:

\[
e_{EM} = 0.0012, \quad e_{Milstein} = 0.0261, \quad e_{RKIV} = 0.0161, \quad e_{RK1.0} = 0.0311
\]

respectively. The relative error for Wong-Zakai method is obtained as \( e_{WZ} = 0.0100 \), This result suggests that Wong-Zakai method performs better than the strong order 1.0 Runge-Kutta, Runge-Kutta IV and Milstein methods for problem 3. All of the relative errors have been shown below for the methods in the figure (Fig. 5) along with the solution graphs of the corresponding methods (Fig. 6).

Table 3. Comparison of results for the corresponding methods in problem 3

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It should be underlined that the comparison can be made for any time point \( t_i \) for these methods and the results could be obtained differently for varying time steps \( h \).
Fig. 3. Error plots for the methods in problem 2

Fig. 4. Solution plots for the methods in problem 2
If the simulations are carried out with different values for $h$, for instance with $h = 0.025$, the relative errors are obtained at $t = 3.0$ as:
\[ e_{EM} = 0.0065, \ e_{Mil\text{stein}} = 0.0269, \ e_{RKIV} = 0.0092, \ e_{RK1.0} = 0.0081 \]

respectively. The relative error for Wong-Zakai method is obtained as \( e_{WZ} = 0.0154 \). At \( t = 1.5 \), these values are obtained as:

\[ e_{EM} = 0.0025, \ e_{Mil\text{stein}} = 0.0202, \ e_{RKIV} = 0.0019, \ e_{RK1.0} = 0.0013 \]

respectively. The relative error for Wong-Zakai method is obtained as \( e_{WZ} = 0.0028 \). For a different number of simulations, for instance with \( N = 200000 \), the relative errors at \( t = 3.0 \) are obtained as

\[ e_{EM} = 0.0084, \ e_{Mil\text{stein}} = 0.0380, \ e_{RKIV} = 0.0046, \ e_{RK1.0} = 0.0219 \]

respectively. The relative error for Wong-Zakai method is obtained as \( e_{WZ} = 0.0031 \). At \( t = 1.5 \), these values are obtained as:

\[ e_{EM} = 0.0069, \ e_{Mil\text{stein}} = 0.0233, \ e_{RKIV} = 0.0016, \ e_{RK1.0} = 0.0104 \]

respectively. The relative error for Wong-Zakai method is obtained as \( e_{WZ} = 0.0022 \). It is clear that Wong-Zakai performs similarly for varying values of \( h, T \) and \( N \) as well.

### 5 Conclusion

In this study, we have used thee reducible stochastic differential equations with explicit solutions to compare the efficiency of Wong-Zakai method with other frequently used stochastic methods in literature. Wong-Zakai method requires a Stratonovich type stochastic differential equation for implementation, hence the three numerical examples were transformed to Stratonovich type SDEs. Similar time intervals were used for three examples with the same time steps for an accurate comparison of Wong-Zakai method with the strong order 1.0 Runge-Kutta, Runge-Kutta IV, Euler-Maruyama and Milstein methods. The relative errors were analyzed in comparison to the analytical solutions of the problems and it is seen that Wong-Zakai method performs similarly in comparison to the other four popular methods. Hence, it can be said that Wong-Zakai method provides an efficient alternative for the investigation of reducible stochastic differential equations. The method can be generalized to systems of SDEs and stochastic models for further analysis.

### Competing Interests

Authors have declared that no competing interests exist.

### References


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