On Pairwise L-Closed Spaces

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Abstract

In this paper we define pairwise L-closed spaces and study their properties, we obtain several results concerning pairwise L-closed spaces, and some product theorems. Some examples dealing with pairwise L-closed spaces are discussed.

Keywords: Pairwise L-closed space; pairwise P-space; p-Lindelöf space; s-Lindelöf space; p-continuous function; p-homeomorphism function.

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1 Introduction

In mathematics, the notion of bitopological spaces is introduced and studied by J.C Kelly [1] in 1923, he defined pairwise Hausdorff, pairwise regular, pairwise normal spaces, and obtained generalizations of several standard results such as Urysohn’s Lemma and Tietze’s extension theorem.

Since then several mathematicians studied various concepts in bitopological spaces which turned to be an important field in general topology. We use $\mathbb{R}$ and $\mathbb{N}$ to denote the set of all real and natural numbers respectively, $p$- to denote pairwise and $\tau_{\text{cococ}}, \tau_{\text{dis}}, \tau_i, \tau_u, \tau_r$ to denote co-countable, discrete, Sorgenfrey, usual and right ray topologies on $\mathbb{R}$ or $\mathbb{N}$. Also the $\tau_i$-closure of a set $A$ is denoted by $cl_iA$. Also we study the properties of pairwise L-closed spaces and their relations with other related concepts.

2 Preliminaries

**Definition 2.1:** A bitopological space $(X, \tau_1, \tau_2)$ is said to be pairwise L-closed space if each $\tau_1$-Lindelöf subset of $X$ is $\tau_2$-closed and each $\tau_2$-Lindelöf subset of $X$ is $\tau_1$-closed.

**Definition 2.2:** A family $F$ of non-empty $\tau_1$-closed subsets or $\tau_2$-closed subsets of a bitopological space $(X, \tau_1, \tau_2)$ is called a $p$-closed family if it contains at least one member $F_1$ and at least one member $F_2$ such that $F_1$ is $\tau_1$-closed proper subset of $X$ and $F_2$ is $\tau_2$-closed proper subset of $X$. A family $F$ of non-empty subsets of $X$ is $\tau_1\tau_2$-closed if every member of $F$ is $\tau_1$-closed or $\tau_2$-closed (see 2.25 [2]).

**Definition 2.3:** A cover $\tilde{U}$ of a bitopological space $(X, \tau_1, \tau_2)$ is called $\tau_1\tau_2$-open cover if $\tilde{U} \subseteq \tau_1 \cup \tau_2$, and it is called $p$-open cover for $X$ if it contains at least one non empty member of $\tau_1$ and at least one non empty member of $\tau_2$ (see 2.26 [2]).

**Definition 2.4:** A bitopological space $(X, \tau_1, \tau_2)$ is said to be $p$-Lindelöf if every $p$-open cover for $X$ has a countable subcover. Also $X$ is called $s$-Lindelöf if every $\tau_1\tau_2$-open cover for $X$ has a countable subcover (see 2.5 [2]).

**Definition 2.5:** A bitopological space $(X, \tau_1, \tau_2)$ is $\tau_1$-Lindelöf with respect to $\tau_2$ if for each $\tau_1$-open cover for $X$ there is a countable $\tau_1$-open subcover. Now if $X$ is $\tau_1$-Lindelöf with respect to $\tau_2$ and it is $\tau_2$-Lindelöf with respect to $\tau_1$, then $X$ is called B-Lindelöf (see 2.6 [2]).

**Definition 2.6:** A bitopological space $(X, \tau_1, \tau_2)$ is called pairwise T1 if for each pair of distinct points $x, y$ in $X$, there exists a $\tau_1$-neighbourhood $U$ of $x$ and a $\tau_2$-neighbourhood $V$ of $y$ such that $x \in U$, $y \notin U$ and $y \in V, x \notin V$ (see 2.4 [3]).

**Definition 2.7:** A bitopological space $(X, \tau_1, \tau_2)$ is called $p$-Hausdorff if $\forall x \neq y$ in $X$, there exists a $\tau_1$-neighbourhood $U$ of $x$ and a $\tau_2$-neighbourhood $V$ of $y$ such that $x \in U, y \in V$, and $U \cap V = \phi$ (see 2.6 [3]).

**Definition 2.8:** [4] A bitopological space $(X, \tau_1, \tau_2)$ is said to be pairwise P-space if countable intersection of $\tau_1$-open subsets of $X$ is a $\tau_2$-open subset of $X$ and countable intersection of $\tau_2$-open subsets of $X$ is a $\tau_1$-open subset of $X$. A point $x \in X$ is called a P-point if the intersection of countably many $\tau_1$-neighborhoods of $x$ is a $\tau_2$-neighborhood of $x$, and the intersection of countably many $\tau_2$-neighborhoods of $x$ is a $\tau_1$-neighborhood of $x$.

**Definition 2.9:** A bitopological space $(X, \tau_1, \tau_2)$ is called second countable if $(X, \tau_1)$ is second countable and $(X, \tau_2)$ is second countable (see 2.7 [2]).
Definition 2.10: [2] A bitopological space \((X, \tau_1, \tau_2)\) is called Lindelöf (resp. \(\tau_1\)-compact) if it is \(\tau_1\)-Lindelöf (resp. \(\tau_1\)-compact) and \(\tau_2\)-Lindelöf (resp. \(\tau_1\)-compact).

Example 2.11: Consider the bitopological space \((\mathbb{R}, \tau_a, \tau_1)\), let \(A = [0,1]\), then \(A\) is a \(\tau_a\)-closed subset of \(\mathbb{R}\). Furthermore \(A\) is \(\tau_a\)-Lindelöf because \((\mathbb{R}, \phi_a)\) is Lindelöf. Now \(A\) is neither closed nor open in \((\mathbb{R}, \tau_a, \tau_1)\), hence \((\mathbb{R}, \tau_a, \tau_1)\) is not a pairwise \(L\)-closed space.

Proposition 2.12: In a bitopological space \((X, \tau_1, \tau_2)\), if every countable subset of \(X\) is closed, then every countable subset is discrete and every compact subset is finite (see 2.1 [3]).

Corollary 2.13: If \((X, \tau_1, \tau_2)\) is a pairwise \(L\)-closed space, every countable subset of \(X\) is closed, discrete and every compact subset of \(X\) is finite.

Proposition 2.14: Every subspace of a pairwise \(L\)-closed space is pairwise \(L\)-closed.

Proof: Suppose that \((X, \tau_1, \tau_2)\) is a pairwise \(L\)-closed space and \(Y\) is a subspace of it, let \(F\) be a \(\tau_1\)-Lindelöf subset of \(Y\), then \(F\) is a \(\tau_1\)-Lindelöf subset of \(X\), hence \(F\) is a \(\tau_2\)-closed subset of \(X\) because \(X\) is a pairwise \(L\)-closed space. Similarly if we suppose that \(G\) is a \(\tau_1\)-Lindelöf subset of \(Y\), then \(G\) is \(\tau_2\)-closed. Thus \(Y\) is a pairwise \(L\)-closed space.

Corollary 2.15: If \((X, \tau_1, \tau_2)\) is a p-Hausdorff pairwise P-space, then \(X\) is a pairwise \(L\)-closed space.

Proof: Let \(F\) be a \(\tau_1\)-Lindelöf subset of \(X\), let \(x \in X\) such that \(x \notin F\). Since \((X, \tau_1, \tau_2)\) is p-Hausdorff, \(\exists\) a sequence \(w_k\) of \(\tau_1\)-open subsets such that \(x \in \cap_{k=1}^{\infty} w_k\), also \(\exists\) a sequence \(v_k\) of \(\tau_2\)-open subsets such that \(F \subseteq \cap_{k=1}^{\infty} v_k\), and \(w_k \cap v_k = \phi \forall k \in \mathbb{N}\). \(X\) is pairwise P-space, so \(\cap_{k=1}^{\infty} w_k\) is \(\tau_2\)-open subset containing \(x\) and \(\cap_{k=1}^{\infty} v_k \cap F = \phi\), so \(F\) is a \(\tau_2\)-closed subset of \(X\). Similarly if we suppose that \(G\) is a \(\tau_2\)-Lindelöf subset of \(X\), we will get that it is \(\tau_1\)-closed. Hence \(X\) is a pairwise \(L\)-closed space.

Proposition 2.16: Every Lindelöf pairwise \(L\)-closed bitopological space is a pairwise P-space.

Proof: Let \((X, \tau_1, \tau_2)\) be a Lindelöf pairwise \(L\)-closed space, let \(A = \cap_{k=1}^{\infty} u_k\) be a \(\tau_1\)-G\(_4\) set, then \(A\) is a \(\tau_2\)-open subset of \(X\) since \(X - A = X - \cap_{k=1}^{\infty} u_k = \cup_{k=1}^{\infty} (X - u_k)\) is a \(\tau_2\)-P\(_\sigma\)-set, so \(X - A\) is a \(\tau_1\)-Lindelöf subset of \(X\) because \(X\) is Lindelöf, but \(X\) is a pairwise \(L\)-closed space, so \(X - A\) is a \(\tau_2\)-closed subset of \(X\). Hence \(A\) is a \(\tau_2\)-open subset of \(X\). Similarly, if we suppose that \(B\) is a \(\tau_2\)-G\(_4\)-set, we will get that it is a \(\tau_1\)-open subset of \(X\). Thus \(X\) is a pairwise P-space.

Corollary 2.17: For a p-Hausdorff Lindelöf bitopological space \((X, \tau_1, \tau_2)\), \(X\) is a pairwise \(L\)-closed space if and only if it is pairwise P-space.

The proof follows from 2.15 and 2.16.

Definition 2.18: In a bitopological space \((X, \tau_1, \tau_2)\), \(\tau_1\) is regular with respect to \(\tau_2\) if \(\forall x \in X\) and each \(\tau_1\)-closed set \(F\) such that \(x \notin F\), there exists a \(\tau_1\)-open set \(U\) and a \(\tau_2\)-open set \(V\) such that \(x \in U\) and \(F \subseteq V\) and \(U \cap V = \phi\). (see 4.3 [1])

Definition 2.19: [1] A bitopological space \((X, \tau_1, \tau_2)\) is called p-regular if \(\tau_1\) is regular with respect to \(\tau_2\) and \(\tau_2\) is regular with respect to \(\tau_1\).

Definition 2.20: In a bitopological space \((X, \tau_1, \tau_2)\), a point \(x \in X\) has a pairwise \(L\)-closed neighborhood \(U\) if each \(\tau_1\)-Lindelöf subset of \(U\) containing \(x\) is \(\tau_2\)-closed, and each \(\tau_2\)-Lindelöf subset of \(U\) containing \(x\) is \(\tau_1\)-closed.

Proposition 2.21: Let \((X, \tau_1, \tau_2)\) be a p-regular space. If every point in \(X\) has a pairwise \(L\)-closed
neighborhood, then \((X, \tau_1, \tau_2)\) is pairwise L-closed.

Proof: Let \(F\) be a \(\tau_1\)-Lindelöf subset of \(X\), let \(x \in X\) such that \(x \notin F\). If \(U\) is a \(\tau_1\)-open subset containing \(x\), then \(U\) is L-closed neighborhood. Since \(X\) is p-regular, \(\exists a \tau_1\)-open set \(H\) such that \(x \in H \subseteq cl_H \subseteq U\) and \(cl_H \cap F\) is a \(\tau_1\)-Lindelöf subset of \(U\), hence \(cl_H \cap F\) is a \(\tau_2\)-closed subset of \(U\). \(U - cl_H \cap F\) is a \(\tau_2\)-open neighborhood of \(x\), so \(\{U - cl_H \cap F\} \cap F = \emptyset\) is a contradiction. Hence \(x \in F\) and \(F\) is a \(\tau_2\)-closed subset of \(X\). Similarly if we assume that \(G\) is a \(\tau_2\)-Lindelöf subset of \(X\), by a similar argument we will get that \(G\) is \(\tau_1\)-closed. So \((X, \tau_1, \tau_2)\) is a pairwise L-closed space.

**Definition 2.22:** A space \((X, \tau_1, \tau_2)\) is said to be p-normal if for a \(\tau_1\)-closed set \(C\) and a \(\tau_1\)-closed set \(F\) such that \(C \cap F = \emptyset\), there exist a \(\tau_1\)-open set \(G\), a \(\tau_2\)-open set \(V\) such that \(F \subseteq G\), \(C \subseteq V\) and \(V \cap G = \emptyset\). (see 2.18 [5])

**Proposition 2.23:** A p-regular pairwise L-closed space is p-normal.

Proof: Let \((X, \tau_1, \tau_2)\) be a pairwise L-closed space, let \(A\) be a \(\tau_1\)-Lindelöf subset of \(X\) and \(B\) be a \(\tau_2\)-Lindelöf subset of \(X\) such that \(A \cap B = \emptyset\), then \(A\) is \(\tau_2\)-closed and \(B\) is \(\tau_1\)-closed because \(X\) is pairwise L-closed. Since \((X, \tau_1, \tau_2)\) is p-regular we have, \(\forall a \in A, \exists a \tau_1\)-closed subset \(F_a\) and a \(\tau_2\)-open subset \(G_a\) such that \(a \in G_a \subseteq F_a \subseteq X - B\). Now \(\forall b \in B, \exists a \tau_1\)-open subset \(C_b\) and \(\tau_2\)-closed subset \(M_b\) such that \(b \in C_b \subseteq M_b \subseteq X - A\). Let \(\tilde{U} = \{G_a : a \in A\}\) be a \(\tau_2\)-open cover for \(A\) and \(\tilde{V} = \{C_b : b \in B\}\) be a \(\tau_1\)-open cover for \(B\). \(A\) and \(B\) are \(\tau_1\)-Lindelöf and \(\tau_2\)-Lindelöf respectively, so \(A \subseteq \bigcup_{k=1}^{\infty} G_k\) and \(B \subseteq \bigcup_{k=1}^{\infty} C_k\). Let \(V_1 = C_1\) and for each positive integer \(k > 1\), let \(V_k = C_k - \bigcup_{j=1}^{k-1} F_j\). For each positive integer \(k\), let \(H_k = G_k - \bigcup_{j=1}^{k-1} M_j\) and \(V = \bigcup_{k=1}^{\infty} V_k\), \(H = \bigcup_{k=1}^{\infty} H_k\), then \(V\) is a \(\tau_1\)-open subset of \(X\) and \(H\) is a \(\tau_2\)-open subset of \(X\). Also \(A \subseteq V, B \subseteq H\). Furthermore, if \(x \in H \cap V\), then \(x \in H_i \cap V_j\) for some \(i, j \in \mathbb{N}\), and so \(x \in (G_i - \bigcup_{j=1}^{i-1} M_j) \cap (C_i - \bigcup_{j=1}^{i-1} F_j)\).

Considering separately the cases \(i > j\) and \(i \leq j\) yields a contradiction and so \(H \cap V = \emptyset\). Thus \(X\) is p-normal.

**Definition 2.24:** A bitopological space \((X, \tau_1, \tau_2)\) is said to be pairwise hereditarily Lindelöf if every \(\tau_1\)-subspace of \(X\) is Lindelöf and \(\tau_2\)-subspace of \(X\) is Lindelöf.

**Corollary 2.25:** For a pairwise hereditary Lindelöf bitopological space \((X, \tau_1, \tau_2)\), the following are equivalent:

a. \(X\) is a pairwise L-closed space.

b. \(X\) is a countable discrete space.

**Proposition 2.26:** Every p-regular space which can be represented as a countable union of subspaces each of which has the pairwise L-closeness property has itself the pairwise L-closeness property.

Proof: Suppose that \(X = \bigcup_{k=1}^{\infty} X_k\), \(X_k\) is pairwise L-closed subspace. Let \(A\) be a \(\tau_1\)-Lindelöf subset of \(X_k\) for some \(k \in \mathbb{N}\), then \(A\) is a \(\tau_1\)-Lindelöf subset of \(X\) where \(X\) is p-regular. But \(A\) is a \(\tau_1\)-closed subset of \(X_k\) because \(X_k\) is a pairwise L-closed subspace of \(X\), hence \(A\) is a \(\tau_1\)-closed subset of \(X\) \(\forall i, j = 1, 2\) \(i \neq j\). Thus \(X\) is pairwise L-closed.

**Proposition 2.27:** In the bitopological space \((X, \tau_1, \tau_2)\), the sum \(\bigoplus_{\alpha \in A} X_\alpha\), where \(X_\alpha\) for some \(\alpha \in A\) has a pairwise L-closeness property if and only if all spaces \(X_\alpha\) have a pairwise L-closeness property.

Proof: \(\Rightarrow\) Suppose that the sum \(\bigoplus_{\alpha \in A} X_\alpha\), where \(X_\alpha \neq \emptyset\) for some \(\alpha \in A\) is pairwise L-closed space, then \(X_\alpha\) is a pairwise L-closed subspace of \(\bigoplus_{\alpha \in A} X_\alpha\) because \(X_\alpha\) is a closed subspace of \(\bigoplus_{\alpha \in A} X_\alpha\).
Suppose that $F$ is a $\tau_i$-Lindelöf subset of $\bigoplus_{a \in A} X_a$, then $F \cap X_a$ is a $\tau_i$-Lindelöf subset of $X_a$. But $X_a$ is pairwise $L$-closed, hence $F$ is a $\tau_j$-closed subset of $X_a$, so $F$ is a $\tau_j$-closed subset of $\bigoplus_{a \in A} X_a$ for all $\alpha \in A$, $i, j = 1, 2, i \neq j$. Thus $\bigoplus_{a \in A} X_a$ is pairwise $L$-closed.

**Definition 2.28:** [4] A bitopological space $(X, \tau_1, \tau_2)$ is pairwise almost Lindelöf if every $\tau_1$-open cover $\bigcup \{ U_\alpha : \alpha \in A \}$ of $X$ has a countable subcollection $\bigcup \{ U_\alpha : \alpha \in A_1 \subseteq A \}$ of $A$ such that $X = \bigcup_{\alpha \in A} U_\alpha$. If $(X, \tau_1, \tau_2)$ is pairwise almost Lindelöf, then $X$ is pairwise $L$-closed because $f$ is one to one. Both $X$ is pairwise almost Lindelöf, and $X = \bigcup_{\alpha \in A} U_\alpha$. Hence $F$ is $\tau_j$-closed, and $X = H_1 \cup H_2 \subseteq \cup_{a \in A} H_{u_a}$. But $X$ is pairwise $L$-closed, hence $F$ is $\tau_j$-closed, and $X = H_1 \cup H_2 \subseteq \cup_{a \in A} H_{u_a}$.

**Definition 2.29:** A bitopological space $(X, \tau_1, \tau_2)$ is called pairwise hereditarily almost Lindelöf if every subspace of $X$ is pairwise almost Lindelöf.

**Proposition 2.30:** If $(X, \tau_1, \tau_2)$ is a pairwise $L$-closed space, then the following are equivalent:

a. $X$ is pairwise hereditarily almost Lindelöf.

b. $X$ is pairwise hereditarily Lindelöf.

c. $X$ is countable discrete.

Proof: $c \implies a$ Suppose that $X$ is a countable discrete space such that $X = \bigcup_{a \in A} F_k$, $F_k$ is a $\tau_i$-Lindelöf subset of $X$. If $\bigcup \{ u_\alpha : \alpha \in A \}$ is a $\tau_i$-open cover for $F_k$ and $\bigcup \{ u_\alpha : \alpha \in A_1 \subseteq A \}$ is a countable subcollection of $A$ where $u_\alpha$ is a $\tau_i$-open cover for $X$, then $F_k \subseteq \cup_{a \in A} u_{a_k}$. But $X$ is pairwise $L$-closed, hence $F_k$ is $\tau_j$-closed, and $X = \bigcup_{a \in A} F_k \subseteq \cup_{a \in A} u_{a_k}$ for all $j = 1, 2, i \neq j$. Thus $X$ is pairwise hereditarily almost Lindelöf.

**Proposition 2.31:** Every $\tau_1\tau_2$-open cover for a $p$-regular pairwise $L$-closed space $(X, \tau_1, \tau_2)$ has locally countable open refiment.

Proof: Suppose that $\bigcup \{ U_x : x \in X \}$ is a $\tau_1\tau_2$-open cover for $X$. Since $X$ is $p$-regular, $\forall x \in X$, there exists a $\tau_1$-open set $u_x$ such that $x \in u_x \subseteq c_2 u_x \subseteq u_x$ for some $\tau_1$-open set $u_x$. Let $\{ u_{x_k} : k \in \mathbb{N} \}$ be a countable subcover for $\bigcup \{ U_x : x \in X \}$. The sets $H_k = u_{x_k} - (c_2 u_{x_k} \cup d_2 u_{x_k} \cup \ldots)$ are $\tau_i$-open or $\tau_2$-open that constitute a $\tau_1\tau_2$-open cover for $X$. $\forall x \in X$ we have $x \in H_k(x)$ where $h(x)$ is the smallest integer such that $x \in u_{x_k}$. $\{ H_k : k \in \mathbb{N} \}$ refines $\bigcup \{ U_x : x \in X \}$.

3 Product Properties of Pairwise $L$-closed Spaces

**Definition 3.1:** If $(X, \tau_1, \tau_2)$ and $(Y, \sigma_1, \sigma_2)$ are two bitopological spaces, a function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be $p$-continuous if $f_1 : (X, \tau_1) \rightarrow (Y, \sigma_1)$ is continuous and $f_2 : (X, \tau_2) \rightarrow (Y, \sigma_2)$ is continuous.

**Proposition 3.2:** Let $(X, \tau_1, \tau_2)$ and $(Y, \sigma_1, \sigma_2)$ be two bitopological spaces such that $(Y, \sigma_1, \sigma_2)$ is a pairwise $L$-closed space, if $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is a $p$-continuous one to one function, then $(X, \tau_1, \tau_2)$ is a pairwise $L$-closed space.

Proof: Suppose that $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is a $p$-continuous one to one function. Let $(Y, \sigma_1, \sigma_2)$ be a pairwise $L$-closed space, let $F$ be a $\tau_1$-Lindelöf subset of $X$, then $f(F)$ is $\sigma_2$-Lindelöf because $f$ is a $p$-continuous function, but $(Y, \sigma_1, \sigma_2)$ is a pairwise $L$-closed space, so $f(F)$ is a $\sigma_2$-closed subset of $Y$. Now $F = f^{-1}(f(F))$ is a $\tau_2$-closed subset of $X$ since $f$ is one to one. Similarly if we suppose that $G$ is a $\tau_2$-Lindelöf subset of $X$, we will get that it is $\tau_2$-closed. Hence $(X, \tau_1, \tau_2)$ is pairwise $L$-closed.

**Definition 3.3:** Let $(X, \tau_1, \tau_2)$ and $(Y, \sigma_1, \sigma_2)$ be two bitopological spaces, a function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is called a $p$-homeomorphism if $f$ is a bijection, $p$-continuous and $f^{-1}$ is $p$-continuous. $(X, \tau_1, \tau_2)$
and \((Y, \sigma_1, \sigma_2)\) are called p-homeomorphic.

**Definition 3.4:** A function \(f:(X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)\) is called pairwise open if the induced functions \(f_1:(X, \tau_1) \rightarrow (Y, \sigma_1)\) and \(f_2:(X, \tau_2) \rightarrow (Y, \sigma_2)\) are both open. A function \(f:(X, \tau) \rightarrow (Y, \sigma)\) is closed if it sends closed sets onto closed sets. The function \(f:(X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)\) is called pairwise closed if the induced functions \(f_1:(X, \tau_1) \rightarrow (Y, \sigma_1)\) and \(f_2:(X, \tau_2) \rightarrow (Y, \sigma_2)\) are both closed (see 2.45 [2]).

**Proposition 3.5:** Let \((X, \tau_1, \tau_2)\) be a \(p\)-Lindelöf bitopological space, \((Y, \sigma_1, \sigma_2)\) is a pairwise \(L\)-closed space, if \(f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)\) is a bijection \(p\)-continuous function, then \(f\) is \(p\)-homeomorphism.

**Proposition 3.7:** Pairwise \(L\)-closeness property is a bitopological property.

**Proof:** Let \((X, \tau_1, \tau_2)\) be a pairwise \(L\)-closed space and \((Y, \sigma_1, \sigma_2)\) be any bitopological space. Suppose that \(h: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)\) is \(p\)-homeomorphism, let \(A\) be a \(\tau_1\)-Lindelöf subset of \(X\), then \(h(A)\) is \(\sigma_1\)-Lindelöf because \(h\) is \(p\)-continuous. Since \(X\) is a pairwise \(L\)-closed space, \(A\) is \(\tau_2\)-closed, hence \(h(A)\) is \(\sigma_2\)-closed because \(h\) is pairwise \(L\)-closed. Similarly if we suppose that \(B\) is \(\tau_2\)-Lindelöf, we will get that \(h(B)\) is \(\sigma_1\)-closed. Thus \(Y\) is a pairwise \(L\)-closed space.

**Remark 3.8:** The product of two Lindelöf topological spaces need not to be Lindelöf. In general the product of two Lindelöf bitopological spaces is not necessarily Lindelöf as the following example shows (see 2.21 [2]).

Let \(X = \mathbb{R} \times I\) where \(I\) is an interval, let \(\leq\) be the lexicographical order in \(X\). Let \(\beta_1 = \{[x, y): x < y, x, y \in \mathbb{R}\}\) be a base for the lower limit topology (or Sorgenfrey topology) \(\tau_1\) on \(X\) and \(\beta_2 = \{(x, y]: x < y, x, y \in \mathbb{R}\}\) be a base for \(\tau_2\) on \(X\), so \((X, \tau_1, \tau_2)\) is a Lindelöf bitopological space. \((X \times \tau_1, \tau_1 \times \tau_2)\) is not \((\tau_1 \times \tau_1)\)-Lindelöf because \(\tau_1 \times \tau_1\)-closed subspace \(L = \{(x, y): x = -y, x, y \in \mathbb{R}\}\) is not a \((\tau_1 \times \tau_1)\)-Lindelöf subspace, it is discrete.

**Proposition 3.9:** If \((X, \tau_1, \tau_2)\) and \((Y, \sigma_1, \sigma_2)\) are pairwise \(L\)-closed bitopological spaces such that either \(X\) or \(Y\) is \(p\)-regular, then \(X \times Y\) is a pairwise \(L\)-closed space.

**Proof:** Suppose that \((X, \tau_1, \tau_2)\) and \((Y, \sigma_1, \sigma_2)\) are pairwise \(L\)-closed spaces, let \(Y\) be \(p\)-regular, let \(F\) be a \((\tau_1 \times \sigma_1)\)-Lindelöf subset of \(X \times Y\). If \((x_0, y_0) \notin F\), so \((x_0, y_0) \notin (\{x_0\} \times Y) \cap F\) and \((\{x_0\} \times Y) \cap F\) is a \(\tau_2\)-closed subset of \(X \times Y\) because \(Y\) is pairwise \(L\)-closed. Since \(Y\) is \(p\)-regular,\(\exists\) \(\sigma_1\)-open set \(H\) containing \(y_0\) such that \((X \times cl_2 H) \subseteq (X \times (\{x_0\} \times Y) \cap F)\), so the projection function \(p_x(X \times cl_2 H) \cap ((\{x_0\} \times Y) \cap F)\) is a \(\tau_2\)-closed subset of \(X\) because \(p_x\) is \(p\)-continuous. \(X - (\{x_0\} \times Y) \cap F\) is \(\tau_2\)-open neighborhood of \((x_0, y_0)\) disjoint from \(F\), hence \(F\) is \((\tau_2 \times \sigma_2)\)-closed subset of \(X \times Y\). Similarly if we suppose that \(G\) is a \((\tau_2 \times \sigma_2)\)-Lindelöf subset of \(X \times Y\), then it is a \((\tau_1 \times \sigma_1)\)-closed subset of \(X \times Y\). Therefore \(X \times Y\) is pairwise \(L\)-closed.

**Proposition 3.10:** The product of two finite number of pairwise \(L\)-closed \(p\)-regular spaces is pairwise \(L\)-closed.
Proof: Let \{X_k : k \in \mathbb{N}\} be a family of finitely many p-regular pairwise L-closed spaces. Let \(X = X_k\), by induction on \(k\), for \(k = 2\) the result is given by 3.9. Suppose that the result is true for \(k = n\) \(\forall n \in \mathbb{N}\), we want to show that it is true for \(k = n + 1\). Now \((X_1 \times X_2 \times \ldots \times X_n) \times X_{n+1}\) is p-homeomorphic to \(X_1 \times X_2 \times \ldots \times X_n \times X_{n+1}\), so by induction hypothesis we get that \(X_1 \times X_2 \times \ldots \times X_n \times X_{n+1}\) is pairwise L-closed. Hence \(X\) is a pairwise L-closed.

**Definition 3.11:** A surjective function \(f(X, \tau) \rightarrow (Y, \sigma)\) is a Lindel" of function if whenever \(K\) is a Lindel" of closed subset of \(Y\), we have \(f^{-1}(K)\) is a Lindel" of subset of \(X\). A surjective function \(f(X, \tau_1, \tau_2) \rightarrow (Y_1, \sigma_1, \sigma_2)\) is called pairwise Lindel" of function if the induced function \(f(X, \tau_i) \rightarrow (Y_i, \sigma_i)\) is Lindel" of function \(\forall i = 1, 2\).

**Proposition 3.12:** Let \((X, \tau_1, \tau_2)\) be a pairwise L-closed space, and \((Y, \sigma_1, \sigma_2)\) be a Lindel" of space, then \(\pi_x : X \times Y \rightarrow X\) is a pairwise Lindel" of function.

Proof: Let \(F\) be a \(\tau_1\)-Lindel" of subset of \(X\), then \(F\) is a \(\tau_2\)-closed subset of \(X\) because \(X\) is pairwise L-closed. The projection function \(\pi_x|_{F \times Y}\) is pairwise-closed such that \((\pi_x|_{F \times Y})^{-1}(x)\) is \(\tau_1\)-Lindel" of because \(\pi_x\) is p-continuous. Similarly if we suppose that \(G\) is \(\tau_2\)-Lindel" of, we will get that \((\pi_x|_{F \times Y})^{-1}(x)\) is \(\tau_2\)-Lindel" of. Hence Lindel" of is a pairwise Lindel" of function.

**Proposition 3.13:** Let \((X, \tau_1, \tau_2)\) be a pairwise L-closed space and \((Y, \sigma_1, \sigma_2)\) be any bitopological space. If \(f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)\) is any pairwise function and \(\{(x, f(x)) : x \in X\}\) is \(p\)-Lindel" of, then \(f\) is \(p\)-continuous.

Proof: Let \(\pi_x\) and \(\pi_y\) be two projection functions, then \(X\) and \(f(X)\) are two Lindel" of sets under \(\pi_x\) and \(\pi_y\). Let \(\pi_{x'} = \pi_{x|f}\), then \(\pi_{x'}\) is a pairwise closed projection function \((1)\) and this is because if \(A \subseteq f(X)\) is \(\tau_1\)-closed subset, then \(A\) is \(\tau_1\)-Lindel" of where \(f(X)\) is Lindel" of \(\forall i, j = 1, 2\) \(i \neq j\). So \(\pi_{x'}(A)\) is a \(p\)-Lindel" of \(p\)-closed because \(X\) is a pairwise L-closed space. Since \(f\) is defined on \(X\), \(\pi_{x'}\) is a bijection function \((2)\). From \(1\) and \(2\) we get \(\pi_{x'}\)-open set \(v \subseteq f\) we have \(\pi_{x'}(v)\) is \(\tau_1\)-open in \(X\). Hence \(f = \pi_y \circ (\pi_{x'}|f)^{-1}\) is \(p\)-continuous.

**Proposition 3.14:** If \((X, \tau_1, \tau_2)\) and \((Y, \sigma_1, \sigma_2)\) are \(p\)-Hausdorff pairwise L-closed spaces, then \((X \times Y, \tau_1 \times \sigma_1, \tau_2 \times \sigma_2)\) is a \(\tau_1 \times \sigma_1\)-L-closed space \(\forall i = 1, 2\).

**Definition 3.15:** Let \((X, \tau)\) be a topological space and \(A \subseteq X\). If for every neighborhood \(U_x\) of \(x \in X\) we have \(|U_x \cap A| = |A|\), then \(x\) is called a complete accumulation point of \(A\).

**Proposition 3.16:** If \((X, \tau_1, \tau_2)\) is a pairwise L-closed space and \(A\) is a \(\tau_1\)-Lindel" of subset of \(X\) such that \(|A| = \omega_1\) \(\forall i = 1, 2\), if \(x\) is an accumulation point of \(A\), then \(x\) is a complete accumulation point.

Proof: Let \(A\) be a \(\tau_1\)-Lindel" of subset of \(X\) such that \(|A| = \omega_1\), then \(A\) is \(\tau_1\)-Lindel" of-closed \(\forall i, j = 1, 2\) \(i \neq j\) because \(X\) is pairwise L-closed. Let \(x\) be an accumulation point of \(A\), hence \(x \in cl_{\tau_1}A = A\). Let \(O_x\) be a \(\tau_1\)-Lindel" of-neighborhood of \(x\), if we take \(f.A \cap O_x \rightarrow A\) defined by \(f(x) = x\), then \(f\) is a \(p\)-continuous one to one function. Hence \(|O_x \cap A| = |A|\).

**Competing Interests**

Authors have declared that no competing interests exist.

**References**


