Aspects of the Fourier-Stieltjes Transform of $C^*$-algebra Valued Measures

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Authors’ contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

This paper deals with the Fourier-Stieltjes transform of $C^*$-algebra valued measures. We construct an involution on the space of such measures, define their Fourier-Stieltjes transform and derive a convolution theorem.

Keywords: $C^*$-algebra; vector measure; Fourier-Stieltjes transform; convolution.

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1 Introduction

Banach space valued measures play an important rôle in the geometric theory of Banach spaces. For instance in [1] the author used the theory of vector measures to prove that $L^1[0,1]$ is not isomorphic to a dual of a Banach space. See [2] for interesting historical notes. It is natural to think that $C^*$-algebra valued measures may be useful in the theory of $C^*$-algebras. This paper is in some manner a contribution in that direction. Here we are interested in the bounded $C^*$-algebra valued measures and their Fourier-Stieltjes transform.

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The rest of the paper is structured as follows. In Section 2, we present basic elements of the theory of \( C^* \)-algebras with examples. In Section 3, we construct an involution on the space of bounded \( C^* \)-algebra valued measures on a locally compact group and finally in Section 4, we defined the Fourier-Stieltjes transform and we prove a convolution theorem.

2 \( C^* \)-algebras: Definition and Examples

In this section, we recall what is a \( C^* \)-algebra and we give various examples. Interested readers can consult [3, 4]. All the vector spaces considered here are complex vector spaces.

**Definition 2.1.** A Banach algebra is a Banach space \( \mathfrak{A} \) which is also an algebra such that
\[
\forall a, b \in \mathfrak{A}, \|ab\| \leq \|a\|\|b\|. \tag{2.1}
\]

**Definition 2.2.** An involution on an algebra \( \mathfrak{A} \) is a map \( \ast: \mathfrak{A} \to \mathfrak{A} \) such that
\[
(a^*)^* = a, \quad (a + b)^* = a^* + b^*, \quad (ab)^* = b^*a^*, \quad (\lambda a)^* = \overline{\lambda}a^*.
\]
for \( a, b \in \mathfrak{A} \) and \( \lambda \in \mathbb{C} \). A \( \ast \)-Banach algebra is a Banach algebra with an involution.

**Definition 2.3.** A \( C^* \)-algebra is a \( \ast \)-Banach algebra \( \mathfrak{A} \) such that for all \( a \in \mathfrak{A} \),
\[
\|a^*a\| = \|a\|^2. \tag{2.2}
\]

The following result is well known as the "\( C^* \)-condition".

**Proposition 2.1.** A \( \ast \)-Banach algebra \( \mathfrak{A} \) in which \( \forall a \in \mathfrak{A}, \|a\|^2 \leq \|a^*a\| \) is a \( C^* \)-algebra.

Let us give some examples of \( C^* \)-algebras.

**Example 2.1.**
1. The set of complex numbers \( \mathbb{C} \) is the prototype of \( C^* \)-algebras. The norm is the modulus \( |z| \) and the \( \ast \)-operation is the conjugation \( \overline{z} \).
2. Let \( \mathcal{H} \) be a complex Hilbert space. Denote by \( B(\mathcal{H}) \) the set of bounded operators on \( \mathcal{H} \). Then \( B(\mathcal{H}) \) is a \( C^* \)-algebra under the norm
\[
\|T\| = \sup\{\|T\xi\| : \|\xi\| \leq 1\}
\]
and the involution \( T \to T^\ast \) where \( T^\ast \) is the adjoint of \( T \) defined by
\[
\forall \xi, \eta \in \mathcal{H}, \langle T\xi, \eta \rangle = \langle \xi, T^\ast \eta \rangle.
\]
3. Let \( M_n(\mathbb{C}) \) be the set of square complex matrices of order \( n \). It is a \( C^* \)-algebra under the matrix operations, the norm defined by
\[
\|A\| = \left( \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}}
\]
where \( A \) is the matrix \( A = (a_{ij})_{1 \leq i \leq n, 1 \leq j \leq n} \), and the \( \ast \)-operation \( A^\ast = \overline{A} \).
4. Let $X$ be a compact Hausdorff space. Consider $C(X)$ the set of complex continuous functions on $X$. Then $C(X)$ is a $C^*$-algebra under the usual pointwise operations on $C(X)$, the norm defined by

$$\|f\|_\infty = \sup\{|f(x)| : x \in X\}$$

and the $*$-operation

$$f^*(x) = \overline{f(x)}.$$

Now for a locally compact Hausdorff space $X$ one may consider the set $C_0(X)$ instead of $C(X)$ where $C_0(X)$ is the set of complex continuous functions on $X$ that vanish at infinity. Then $C_0(X)$ is a $C^*$-algebra under the same operations, the same norm and the same involution as $C(X)$.

3 A $*$-Banach Algebra Structure on $\mathcal{M}^1(G, \mathfrak{A})$

Here we would like to trace how far the $C^*$ algebraic structure can infer the structure of the space of vector measures on a locally compact group $G$. Let $G$ be a locally compact group and let $\mathfrak{A}$ be a $C^*$-algebra. We denote by $B(G)$ the $\sigma$-field of Borel subsets of $G$. Following [2] we call a vector measure any set function $m : B(G) \to \mathfrak{A}$ such that for any sequence $(A_n)_{n \geq 1}$ of pairwise disjoint elements of $B(G)$ one has

$$(3.1)$$

$$m(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} m(A_n).$$

A vector measure $m$ is said to be bounded if there exists $M > 0$ such that

$$\forall A \in B(G), \|m(A)\| \leq M.$$  

The set of such bounded vector measures is denoted by $\mathcal{M}^1(G, \mathfrak{A})$. The variation of a vector measure $m$ is the set function $|m|$ defined by

$$|m|(A) = \sup \sum_n \|m(A_n)\|,$$

where the supremum is taken over all the partitions $\pi$ of $A$ into pairwise disjoint measurable subsets of $A$. If $|m|(G) < \infty$ then $m$ is called a vector measure of bounded variation. To be concrete let us give an example of a vector measure taken from [2] and adapted to the case of a locally compact group.

**Example 3.1.** We take $G = \mathbb{R}^d$ and we obviously denote by $L^1(\mathbb{R}^d)$ and $C_0(\mathbb{R}^d)$ the Lebesgue space of complex integrable functions on $\mathbb{R}^d$ and the space of complex continuous functions on $\mathbb{R}^d$ which vanish at infinity respectively. The Fourier transform of $f \in L^1(\mathbb{R}^d)$ is

$$(3.2)$$

$$\mathcal{F}f(x) := \widehat{f}(x) := \int_{\mathbb{R}^d} f(t)e^{-i\langle x, t \rangle}dt, x \in \mathbb{R}^d.$$  

The function $\hat{f}$ is a member of $C_0(\mathbb{R}^d)$ and

$$\|\hat{f}\|_\infty \leq \|f\|_1.$$  

Now let $T : L^1(\mathbb{R}^d) \to C_0(\mathbb{R}^d)$ be a bounded linear operator. A concrete example for $T$ is for instance the Fourier transform $\mathcal{F}$ on $\mathbb{R}^d$. Define

$$(3.4)$$

$$m(A) = T(\chi_A)$$

where $\chi_A$ is the characteristic function of $A$.
where $A$ is a member of the Borel $\sigma$-algebra of $G$. Then $\|m(A)\|_{\infty} \leq \|T\|\mu(A)$ where $\mu$ is the Lebesgue measure of $\mathbb{R}^d$. First notice that $m$ is finitely additive. In fact if $A$ and $B$ are disjoint measurable sets then

$$m(A \cup B) = T(\chi_{A \cup B}) = T(\chi_A + \chi_B) = T(\chi_A) + T(\chi_B) = m(A) + m(B).$$  \hspace{1cm} (3.5)

Therefore, for a sequence $(A_n)_{n \geq 1}$ of pairwise disjoint measurable sets we have

$$\|m(\bigcup_{n=1}^{\infty} A_n) - \sum_{n=1}^{k} m(A_n)\| = \|m^{(k+1)}(A) + m(\bigcup_{n=k+1}^{\infty} A_n) - \sum_{n=1}^{k} m(A_n)\|$$

$$\leq \|T\|\mu(A_n)$$

$$= \|T\| \sum_{n=k+1}^{\infty} \mu(A_n) \to 0 \text{ when } k \to \infty$$

since the real series $\sum_{n} \mu(A_n)$ is convergent and therefore the remainder $\sum_{n=k+1}^{\infty} \mu(A_n)$ goes to 0 whenever $k$ tends to $\infty$. We conclude that $m$ is a vector measure taking values in the $C^*$-algebra $C_0(\mathbb{R}^d)$.

To move forward, we present some properties of $\mathcal{M}^1(G, \mathfrak{A})$.

On $\mathcal{M}^1(G, \mathfrak{A})$, one defines the norm:

$$\|m\| = |m|(G)$$  \hspace{1cm} (3.6)

and the convolution product

$$m_1 * m_2(f) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(xy)dm_1(x)dm_2(y),$$  \hspace{1cm} (3.7)

where $m_1, m_2 \in \mathcal{M}^1(G, \mathfrak{A})$ and $f \in C_0(G, \mathfrak{A})$. And one has

$$\|m_1 * m_2\| \leq \|m_1\|\|m_2\|.$$ 

It is well-known that $(\mathcal{M}^1(G, \mathfrak{A}), \| \cdot \|, *)$ is a Banach algebra.

**Proposition 3.1.** If $\mathfrak{A}$ is unital then so is $\mathcal{M}^1(G, \mathfrak{A})$.

**Proof.** Let us assume that $\mathfrak{A}$ has a unit $1_{\mathfrak{A}}$. For $A \in \mathcal{B}(G)$, set

$$\Delta(A) = \delta(A)1_{\mathfrak{A}} = \begin{cases} 1_{\mathfrak{A}} & \text{if } e \in A \\ 0 & \text{otherwise} \end{cases}$$

where $\delta$ is the Dirac mass at $e$ (the neutral element in the group $G$). It follows that

$$\Delta * m(f) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(xy)d\Delta(x)dm(y) = \int_{\mathbb{R}^d} f(y)dm(y) = m(f),$$

that is $\Delta * m = m$. We have also

$$m * \Delta(f) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(xy)d\Delta(x)dm(y) = \int_{\mathbb{R}^d} f(x)dm(x) = m(f),$$

that is $m * \Delta = m$. Hence $\Delta$ is the unit of $\mathcal{M}^1(G, \mathfrak{A})$. \hfill \Box

**Proposition 3.2.** $\mathcal{M}^1(G, \mathfrak{A})$ is an involutive Banach algebra.
We know already that $M^1(G, \mathfrak{A})$ is a Banach algebra. On this algebra, let us now define an involution. For $m \in M^1(G, \mathfrak{A})$, set

$$m^*(A) = m(A^{-1})^*, \forall A \in B(G).$$

(3.8)

where $A^{-1} = \{x^{-1} : x \in A\}$, or equivalently

$$m^*(f) = \int_G f(x^{-1})dm^*(x)$$

(3.9)

where $*$ is the involution of the $C^*$-algebra $\mathfrak{A}$ and $f$ belongs to $C_*(G; \mathfrak{A})$, the space of $\mathfrak{A}$-valued functions with compact support. One can easily check that the mapping $m \mapsto m^*$ defines an involution on $M^1(G, \mathfrak{A})$.

\[ \square \]

4 The Fourier-Stieltjes Transform

Research on the Fourier-Stieltjes transform stays flourishing. A recent study concerning this subject can be found in [5]. Our analysis here borrows ideas from [6, 7, 8, 9]. Methods there were applied to the case where $G$ is a compact group or $G$ acts on a finite dimensional Hilbert $C^*$-module. With a little adaptation we applied it to the case of a general locally compact group. For more informations about representation theory and Fourier analysis on groups, on may consult [10, 11, 12].

There are various formulations of the Fourier-Stieltjes transform depending on the nature of the underlying group and the structure of the codomain of the measures.

In the case $G$ is abelian, the Fourier-Stieltjes transform of the vector measure $m$ is

$$\hat{m}(\chi) = \int_G \overline{\chi(x)}dm(x),$$

(4.1)

where $\chi$ designates a character of the group $G$. If $G$ is compact and $\mathfrak{A} = \mathbb{C}$, then the Fourier-Stieltjes transform of $m$ is a family $(\hat{m}(\sigma))_{\sigma \in \hat{G}}$ of endomorphisms $\hat{m}(\sigma) : \mathcal{H}_\sigma \to \mathcal{H}_\sigma$ given by the relation:

$$\langle \hat{m}(\sigma)\xi, \eta \rangle = \int_G \langle \sigma(x^{-1})\xi, \eta \rangle dm(x), \xi, \eta \in \mathcal{H}_\sigma.$$  

(4.2)

where $\sigma$ is a member of a class of unitary irreducible representation of $G$, $\mathcal{H}_\sigma$ is the representation space of $\sigma$ and $\hat{G}$ is the unitary dual of $G$. When the group $G$ is compact and $\mathfrak{A}$ is a Banach space, the Fourier-Stieltjes transform of a bounded vector measure $m$ on $G$ is defined and studied in [6]. It is interpreted as a family $(\hat{m}(\sigma))_{\sigma \in \hat{G}}$ of sesquilinear mappings $\hat{m}(\sigma) : \mathcal{H}_\sigma \times \mathcal{H}_\sigma \to \mathfrak{A}$ given by:

$$\hat{m}(\sigma)(\xi, \eta) = \int_G \langle \sigma(x^{-1})\xi, \eta \rangle dm(x).$$

(4.3)

We denote the conjugate space of $\mathcal{H}_\sigma$ by $\overline{\mathcal{H}}$. We denote by $\mathcal{H}_\sigma \hat{\otimes} \overline{\mathcal{H}}$ the completion of the normed tensor product space $\mathcal{H}_\sigma \otimes \overline{\mathcal{H}}$ with respect to the projective tensor norm $\pi$. See [13] for more informations on the tensor product of Banach spaces.

Let $m$ be a vector measure on a locally compact group $G$. From [8] we see that the Fourier-Stieltjes transform of $m$ is the collection $(\hat{m}(\sigma))_{\sigma \in \hat{G}}$ of operators $\hat{m}(\sigma) : \mathcal{H}_\sigma \hat{\otimes} \overline{\mathcal{H}} \to \mathfrak{A}$ where each $\hat{m}(\sigma)$ is defined by the integral

$$\hat{m}(\sigma)(\xi \otimes \eta) = \int_G \langle \sigma(x^{-1})\xi, \eta \rangle dm(x).$$

(4.4)

We denote by $\mathcal{L}(\mathcal{H}_\sigma \hat{\otimes} \overline{\mathcal{H}}, \mathfrak{A})$ the set of bounded operators from $\mathcal{H}_\sigma \hat{\otimes} \overline{\mathcal{H}}$ into $\mathfrak{A}$.
Example 4.1. Consider the matrix group \( G = SU(2) \) where

\[
SU(2) = \{ A \in M_2(\mathbb{C}) : A^*A = I, \det A = 1 \} = \begin{cases} a & b \\ -\overline{b} & \overline{a} \end{cases} : a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1.
\]

Let \( H_2 \) be the set of homogeneous polynomials of degree 2 in two variables \( z_1, z_2 \). Then

\[
H_2 = \mathbb{C}z_1^2 \oplus \mathbb{C}z_1z_2 \oplus \mathbb{C}z_2^2.
\]

Now consider the representation \( \sigma : SU(2) \to GL(H_2) \) given by

\[
|\sigma(A)f|(z_1, z_2) = f((z_1, z_2)A), A \in SU(2), f \in H_2.
\]

Consider a bounded linear operator \( T : L^1(SU(2)) \to C_0(SU(2)) \) and the vector measure \( m \) given by \( m(E) = T(\chi_E) \), so that \( m(f) = T f \) for \( f \) integrable with respect to the Haar measure on \( SU(2) \).

Then the Fourier-Stieltjes transform of \( m \) is given by

\[
\hat{m}(\sigma)(f \otimes g) = m(\phi_{f,g}^\sigma) = T(\phi_{f,g}^\sigma)
\]

where \( \phi_{f,g}^\sigma(A) = \langle \sigma(A^{-1})f, g \rangle \).

Proposition 4.1. If \( m \in M^1(G, \mathfrak{A}) \) and \( \sigma \in \widehat{G} \) then \( \hat{m}(\sigma) \in L(\mathcal{H}_\sigma \hat{\otimes}_\sigma \mathcal{P}_\sigma, \mathfrak{A}) \) and \( \| \hat{m}(\sigma) \|_{\mathcal{H}_\sigma \hat{\otimes}_\sigma \mathcal{P}_\sigma \rightarrow \mathfrak{A}} \leq \| m \| \).

Proof. Let \( m \in M^1(G, \mathfrak{A}) \). For each \( \sigma \in \widehat{G} \), we have

\[
\| \hat{m}(\sigma)(\xi \otimes \eta) \| = \| \int_G (\sigma(x^{-1})\xi, \eta)dm(x) \|
\leq \int_G \| (\sigma(x^{-1})\xi, \eta) \|dm(x)
\leq \| \xi \| \| \eta \| \| m \| (G) = \| \xi \| \| \eta \| \| m \| .
\]

Thus \( \hat{m}(\sigma) \) is a bounded operator and \( \| \hat{m}(\sigma) \|_{\mathcal{H}_\sigma \hat{\otimes}_\sigma \mathcal{P}_\sigma \rightarrow \mathfrak{A}} \leq \| m \| . \)

Using arguments from [7, Lemma 4.1.5] applied to the underlying Banach space structure of \( \mathfrak{A} \), one obtains the injectivity of the Fourier-Stieltjes transform \( m \mapsto \hat{m} \).

Proposition 4.2. The map \( m \mapsto \hat{m} \) from \( M^1(G, \mathfrak{A}) \) into \( \prod_{\sigma \in \widehat{G}} L(\mathcal{H}_\sigma \hat{\otimes}_\sigma \mathcal{P}_\sigma, \mathfrak{A}) \) is injective.

Proposition 4.3. If \( m \in M^1(G, \mathfrak{A}) \) and \( T \in L(\mathcal{H}_\sigma \hat{\otimes}_\sigma \mathcal{P}_\sigma, \mathfrak{A}) \) then the mapping

\[
x \mapsto T[(\sigma(x^{-1})\xi) \otimes \eta]
\]

from \( G \) into \( \mathfrak{A} \) is integrable with respect to \( m \).

Proof.

\[
\int_G \| T[(\sigma(x^{-1})\xi) \otimes \eta] \|dm(x) \leq \| T \| \| \xi \| \| \eta \| \int_G \chi_Gdm
= \| T \| \| \xi \| \| \eta \| \| m \| < \infty.
\]

Thus the map \( x \mapsto T[(\sigma(x^{-1})\xi) \otimes \eta] \) is \( m \)-integrable. \( \square \)
For $T \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H}, \mathfrak{A})$ and $m \in \mathcal{M}^1(G, \mathfrak{A})$, one defines the product $\#$ by:

$$T \# [b_m(\sigma)](\xi \otimes \eta) = \int_G T[(\sigma(x^{-1})\xi) \otimes \eta] dm(x). \quad (4.7)$$

Then we have the following analog of the well-known convolution theorem.

**Proposition 4.4.** If $m, n \in \mathcal{M}^1(G, \mathfrak{A})$ then

$$(n \ast m)(\sigma) = \widehat{m}(\sigma) \widehat{n}(\sigma). \quad (4.8)$$

**Proof.** Let $m$ and $n$ be in $\mathcal{M}^1(G, \mathfrak{A})$ and $\xi \otimes \eta \in \mathcal{H} \otimes \mathcal{H}$. We have:

$$[\widehat{m}(\sigma) \widehat{n}(\sigma)](\xi \otimes \eta) = \int_G \widehat{m}(\sigma)[(\sigma(y^{-1})\xi) \otimes \eta] d\nu(y)$$

$$= \int_G \int_G \langle \sigma(x^{-1})\sigma(y^{-1})\xi, \eta \rangle dm(x) d\nu(y)$$

$$= \int_G \int_G \langle \sigma(x^{-1}y^{-1})\xi, \eta \rangle dm(x) d\nu(y)$$

$$= \int_G \int_G \langle (\sigma(y^{-1})\xi) \otimes \eta \rangle dm(y) d\nu(x) \quad \text{(Fubini)}$$

$$= \widehat{n} \ast \widehat{m}(\sigma)(\xi \otimes \eta).$$

Hence

$$\widehat{m}(\sigma) \widehat{n}(\sigma) = (n \ast m)(\sigma). \quad \square$$

**Remark 4.1.** One knows that the convolution product is commutative if and only if the group $G$ is commutative. Thus if $G$ is commutative we have

$$\widehat{m}(\sigma) \widehat{n}(\sigma) = (n \ast m)(\sigma) = (m \ast n)(\sigma).$$

## 5 Conclusion

In this study, we have constructed an involution on the space of bounded measures on a locally compact group taking values in a $C^*$-algebra. The Fourier-Stieltjes transform of a $C^*$-algebra valued measure has been defined and finally a convolution theorem has been proved.

### Competing Interests

Authors have declared that no competing interests exist.

### References


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