On Quasi-Hemi-Slant Riemannian Submersion

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Abstract

We recall the notions of invariant, anti-invariant, semi-invariant, slant, semi-slant, quasi-slant and hemi-slant Riemannian submersions from almost Hermitian manifolds to a Riemannian manifolds. In this paper we contract a Riemannian submersion which generalizes hemi-slant, semi-slant and semi-invariant Riemannian submersions from almost Hermitian manifold to a Riemannian manifold and study its geometry.

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1 Introduction

Immersion and submersion are special tools in differential geometry. They play important roles in Riemannian geometry and other aspects of the differential geometry.

The theory of Riemannian submersion was initiated by O’Neil [1] and Gray [2]. In [3], the Riemannian submersion was considered between almost Hermitian manifold by Watson under the name almost Hermitian submersion in which the Riemannian submersion is an almost complex map and as a result the vertical and horizontal distributions are invariant with respect to the Almost complex structure of the total space of the submersion. In fact Almost hermitian submersion have been actively studied between different kinds of subclasses of almost Hermitian manifolds, for the details see [4]. Almost Hermitian submersion has also been extended to different kinds of sub-classes of almost contact manifolds see [5]. Most of the studies related to Riemannian or almost Hermitian submersion can be found in [4] and [6]. Sahin constructed Gauss-Weingarten-like formulas and introduced the O’Neill’s type tensor fields for Riemannian maps and obtained new conditions for Riemannian maps to be totally geodesic [7]. Sahin also introduced semi-invariant submersions from almost Hermitian manifolds onto Riemannian manifolds as a generalization of holomorphic submersions and anti-invariant submersions [8], and studied the geometry of such maps. He further introduced the notion of slant submersion from almost hermitian manifold onto an arbitrary Riemannian manifold [9]. Different kinds of Riemannian submersions between manifolds endowed with different structures have been studied, examples are the following; Semi-Riemannian submersion and Lorentzian submersion [4], Contact-Complex submersions [10], quaternionic submersion [11], Submersion from Para-Hermitian Manifold [12] etc. It is well known that almost Hermitian submersions have been extended to the almost contact manifolds and quaternion Kähler manifolds (see [4], [10] and [11] for more details and references therein).

In this paper, we introduce a new class of Riemannian submersions which contains hemi-slant, semi-invariant, semi-slant quasi-slant and hemi-slant submersions defined on almost Hermitian manifolds.

We organize the work as follows. In Section 2, we recall some basic definitions and results on invariant, anti-invariant, semi-invariant, slant, semi-slant quasi-slant and hemi-slant submersions introduced in [13], [14], [8], [9] and [15]. In Section 3 we introduce a new class of Riemannian submersions satisfying certain conditions and containing the above listed submersions, while in Section 4 we study the decomposition theorem of the distributions.

2 Prelimineries

Now we give definitions of some Riemannian submersions from an almost hermitian manifold to a Riemannian manifold.

**Definition 2.1.** [15, p. 84] Let π be a Riemannian submersion from an almost Hermitian manifold \((M, g_M, J_M)\) onto a Riemannian manifold \((N, g_N)\). Then we say that π is an invariant Riemannian submersion if the vertical distribution is invariant with respect to the complex structure \(J_M\), i.e,

\[J_M \ker \pi_* = \ker \pi_*\]

From this, we have the following.

**Theorem 2.1.** [15] Let π be a Riemannian submersion from an almost Hermitian manifold \((M, g_M, J_M)\) onto an almost Hermitian manifold \((N, g_N, J_N)\). Then the horizontal distribution is invariant with respect to \(J_M\).
Watson defined almost Hermitian submersions between almost Hermitian manifolds and showed that in most cases, the base manifold and each fiber have the same kind of structure as the total space [3].

**Definition 2.2.** [14] Let $M$ be a complex finite-dimensional almost Hermitian manifold with Hermitian metric $g_M$ and almost complex structure $J$ and $N$ be a Riemannian manifold with Riemannian metric $g_N$. Suppose that there exists a Riemannian submersion $\pi : M \rightarrow N$ such that, $J\ker\pi_\ast \subseteq (\ker\pi_\ast)^\perp$. Then we say that $\pi$ is an anti-invariant Riemannian submersion.

First of all, since the distribution $\ker\pi_\ast$ is integrable, the above definition implies that the integral manifold $\pi^{-1}(q)$, $q \in N$ of $\ker\pi_\ast$ is a totally submanifold of $M$. From Definition 2.2, we have $J(\ker\pi_\ast)^\perp \cap \ker\pi_\ast \neq \{0\}$. We denote the complementary orthogonal subbundle to $J\ker\pi_\ast$ in $(\ker\pi_\ast)^\perp$ by $\nu$. Then we have [15]

$$(\ker\pi_\ast)^\perp = J\ker\pi_\ast \oplus \nu.$$  

It is easy to see that $\nu$ is an invariant subbundle of $(\ker\pi_\ast)^\perp$ with respect to the complex structure $J$, i.e. This implies, for any $Z \in \Gamma((\ker\pi_\ast)^\perp)$,

$$JZ = BZ + CZ,$$

where $BZ \in \Gamma(\ker\pi_\ast)$ and $CZ \in \Gamma(\nu)$. If $\nu = \{0\}$, then $\pi$ is called a Lagrangian submersion.

**Definition 2.3.** [8] Let $\pi$ be a Riemannian map from an almost Hermitian manifold $(M, g_M, J)$ to a Riemannian manifold $(N, g_N)$. Then we say that $\pi$ is a semi-invariant Riemannian map if there is a distribution $D_1 \subseteq \ker\pi_\ast$ such that

$$\ker\pi_\ast = D_1 \oplus D_2,$$

and

$$JD_1 = D_1, \quad JD_2 \subseteq (\ker\pi_\ast)^\perp,$$

where $D_2$ is orthogonal complementary to $D_1$ in $\ker\pi_\ast$.

Let $\nu$ denote the complementary orthogonal subbundle to $J\ker\pi_\ast$ in $(\ker\pi_\ast)^\perp$. Then we have

$$(\ker\pi_\ast)^\perp = JD_2 \oplus \nu.$$  

Obviously $\nu$ is an invariant subbundle of $(\ker\pi_\ast)^\perp$ with respect to the complex structure $J$.

**Definition 2.4.** [9] Let $\pi$ be a Riemannian submersion from an almost Hermitian manifold $(M, g_M, J)$ onto a Riemannian manifold $(N, g_N)$. If for any non-zero vector $X \in \ker\pi_\ast p$, $p \in M$, the angle $\theta(X)$ between $JX$ and the space $\ker\pi_\ast p$ is constant, i.e., it is independent of the choice of the point $p \in M$ and choice of the tangent vector $X$ in $\ker\pi_\ast p$, then we say that $\pi$ is a slant submersion. In this case, the angle $\theta$ is the slant angle of the slant submersion.

It follows from the above definition that the fibers of a slant submersion are slant submanifold of $M$. If the slant angle is $0 < \theta < \pi$ then the the submersion is called a proper slant submersion.

Many examples of slant submersion can be found in [15] and references therein. Here we have the following.

If $\pi$ is a slant submersion from a Kähler manifold $(M, g_M, J)$ onto a Riemannian manifold $(N, g_N)$, then for any $X \in \Gamma((\ker\pi_\ast)$, we write

$$JX = \phi X + \omega X,$$

where $\phi X$ and $\omega X$ are vertical and horizontal parts of $JX$, respectively.
Also for any $Z \in \Gamma((\ker \pi_\ast)^\perp)$, we have
\[ JZ = BZ + CZ, \]
where $BZ$ and $CZ$ are vertical and horizontal components of $JZ$, respectively (see [9] for more detail).

**Definition 2.5.** ([13, 15]) Let $(M, g_M, J)$ be an almost Hermitian manifold and $(N, g_N)$ be a Riemannian manifold. A Riemannian submersion $\pi : (M, g_M, J) \rightarrow (N, g_N)$ is called a semi-slant submersion if there is a distribution $D_1 \subset \ker \pi_\ast$ such that
\[ \ker \pi_\ast = D_1 \oplus D_2, \quad J(D_1) = D_1, \]
and the angle $\theta = \theta(X)$ between $JX$ and the space $(D_2)_p$ is constant for non-zero $X \in D_2$, where $D_2$ is the orthogonal complement of $D_1$ in $\ker \pi_\ast$.

Let $\pi : (M, g_M, J) \rightarrow (N, g_N)$ be a semi-slant submersion. Then for any $X \in \Gamma(\ker \pi_\ast)$, we have
\[ X = PX + QX, \]
where $PX \in \Gamma(D_1)$ and $QX \in \Gamma(D_2)$. Applying $J$ to this equation yields
\[ JX = \phi X + \omega X, \]
where $\phi X = JPX \in \Gamma(\ker \pi_\ast)$ and $\omega X = JQX \in \Gamma((\ker \pi_\ast)^\perp)$. For any $U \in \Gamma((\ker \pi_\ast)^\perp)$, we write
\[ U = \mathcal{V}U + \mathcal{H}U, \]
where $\mathcal{V}U \in \Gamma(\ker \pi_\ast)$ and $\mathcal{H}U \in \Gamma((\ker \pi_\ast)^\perp)$. Then we have
\[ (\ker \pi_\ast)^\perp = \omega D_2 \oplus \mu, \]
where $\mu$ is the orthogonal complement of $\omega D_2$ in $(\ker \pi_\ast)^\perp$ and is invariant under $J$. For more details on semi-slant submersion see [15] and references therein.

### 3 Quasi-Hemi-Slant Submersions

Here we introduce a new Riemannian submersion which generalizes hemi-slant, semi-slant and semi-invariant Riemannian submersions from almost Hermitian manifold to a Riemannian manifold. We start with the following example;

Let $\pi : \mathbb{R}^{12} \rightarrow \mathbb{R}^6$ be a Riemannian map defined by
\[ \pi(x_1, x_2, ..., x_{12}) = (x_2 + x_1, x_3 + x_4, x_6 \cos \theta - x_8 \sin \theta, x_5 - x_7, x_{10} - x_{11}, x_9 + x_{12}), \]
with $\theta \in (0, \frac{\pi}{2})$. Its Jacobian matrix is given by
\[
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cos \theta & 0 & -\sin \theta & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1
\end{pmatrix}
\]
This matrix has rank 6 and therefore $\pi$ is a submersion. The vector fields
\[
X_1 = \partial x_2 - \partial x_1, \quad X_2 = \partial x_3 - \partial x_4, \quad X_3 = \partial x_6 \sin \theta + \partial x_8 \cos \theta,
\]
\[
X_4 = \partial x_5 + \partial x_7, \quad X_5 = \partial x_{10} + \partial x_{11}, \quad X_6 = \partial x_9 - \partial x_{12},
\]
are linearly independent. We can easily see that $\pi_\ast(X_1) = \pi_\ast(X_2) = \ldots = \pi_\ast(X_6) = 0$, i.e.,
\[
\ker \pi_\ast = \text{span}\{X_1, X_2, X_3, X_4, X_5, X_6\}.
\]
Let $Z_1 = \partial x_2 + \partial x_1, \ Z_2 = \partial x_3 + \partial x_4, \ Z_3 = \partial x_6 \sin \theta - \partial x_8 \cos \theta, \ Z_4 = \partial x_5 - \partial x_7, \ Z_5 = \partial x_{10} - \partial x_{11}, \ Z_6 = \partial x_9 + \partial x_{12}$. Moreover,
\[
(\ker \pi_\ast)^\perp = \text{span}\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6\}.
\]
Let $J$ be a $(1,1)$ almost complex structure on $\mathbb{R}^{12}$ such that
\[
JX_1 = -X_2, \quad JX_2 = X_1, \quad JX_3 = -Z_3, \quad JX_4 = Z_4, \quad JX_5 = -Z_5, \quad JX_6 = Z_5,
\]
\[
JZ_1 = Z_2, \quad JZ_2 = -Z_1, \quad JZ_3 = X_3, \quad JZ_4 = -X_4, \quad JZ_5 = -X_6, \quad JZ_6 = X_5.
\]
Let $D = \text{span}\{X_1, X_2\}$, $D^\theta = \text{span}\{X_3\}$ and $D^\perp = \text{span}\{X_4, X_5, X_6\}$. Then $JD = D$, the angle between $JD^\theta$ and $D^\theta$ is a constant $\theta$, and $JD^\perp \subseteq (\ker \pi_\ast)^\perp$. Therefore, $\pi$ is a Riemannian submersion such that its fibre is decomposed as $\ker \pi_\ast = D \oplus D^\theta \oplus D^\perp$.

Using this example, we remark the following.

**Theorem 3.1.** Let $M$ be a $2m$-dimensional almost Hermitian manifold with $g_M$ a Riemannian metric on $M$ and almost complex structure $J$, and $N$ be a Riemannian manifold with Riemannian metric $g_N$. Then there is a Riemannian submersion $\pi : (M, g_M, J) \rightarrow (N, g_N)$ such that its vertical distribution $\ker \pi_\ast$ admits three orthogonal distributions $D$, $D^\theta$ and $D^\perp$ which are invariant, slant and anti-invariant respectively, i.e.,
\[
\ker \pi_\ast = D \oplus D^\theta \oplus D^\perp,
\]
with $JD = D$, the angle $\theta$ between $JD^\theta$ and $D^\theta$ being constant and $JD^\perp \subseteq (\ker \pi_\ast)^\perp$.

If we denote the dimension of $D$, $D^\theta$ and $D^\perp$ by $m_1$, $m_2$ and $m_3$, respectively, then we easily see the following particular cases:

(a) If $m_1 = 0$, then $M$ is a Hemi-slant submersion [16].

(b) If $m_2 = 0$, then $M$ is a semi-invariant submersion [17].

(c) If $m_3 = 0$, then $M$ is a semi-slant submersion [17].

The submersion in Theorem 3.1 will be called **Quasi-Hemi-Slant (SHS) submersion** and the angle $\theta$ is called the **quasi-hemi-slant angle** of the submersion. This means that a quasi-hemi-slant submersion is a generalization of hemi-slant, semi-invariant and semi-slant submersions.

We say that the quasi-hemi-slant submersion $\pi : (M, g_M, J) \rightarrow (N, g_N)$ is proper if $D \neq \{0\}$, $D^\perp \neq \{0\}$ and $\theta \neq 0, \frac{\pi}{2}$. From the above items, hemi-slant submersions, semi-invariant submersions, and semi-slant submersions are all examples of quasi-hemi-slant submersions.

Let $\pi$ be a quasi-hemi-slant submersion from an almost hermitian manifold $(M, g_M, J)$ onto a Riemannian manifold $(N, g_N)$. Then we have
\[
TM = \ker \pi_\ast \oplus (\ker \pi_\ast)^\perp.
\]
Let us define the projections $P$ and $Q$ on the tangent vectors $TM$ of $M$ by $P : TM \rightarrow \ker \pi_\ast$ and $Q : TM \rightarrow (\ker \pi_\ast)^\perp$, respectively. Now, for any $X \in \Gamma(TM)$,
\[
X = PX + QX.
\]
where $PX \in \Gamma(\ker \pi)$ and $QX \in \Gamma((\ker \pi)^\perp)$. 

Now, for any $X \in \Gamma(\ker \pi)$, the vector field $X$ can be written as

$$X = PX + QX + RX, \quad (3.4)$$

where $P$, $Q$, $R$ are projections of $\ker \pi$ onto $D$, $D^\theta$ and $D^\perp$ respectively. Put

$$JX = \phi X + \omega X, \quad (3.5)$$

where $\phi X \in \Gamma(\ker \pi)$ and $\omega X \in \Gamma((\ker \pi)^\perp)$. From $(3.4)$ and $(3.5)$, we have

$$JX = \phi PX + \omega PX + \phi QX + \omega QX + \phi RX + \omega RX. \quad (3.6)$$

This means that

$$J \ker \pi = D \oplus \phi D^\theta \oplus \omega D^\theta \oplus JD^\perp. \quad (3.7)$$

Since $\omega D^\theta \subseteq (\ker \pi)^\perp$ and $JD^\perp \subseteq (\ker \pi)^\perp$, one obtains

$$(\ker \pi)^\perp = \omega D^\theta \oplus JD^\perp \oplus \nu, \quad (3.8)$$

where $\nu$ is the orthogonal complement of $\omega D^\theta \oplus JD^\perp$ in $(\ker \pi)^\perp$ and it is invariant with respect to $J$. Also, for any $Z \in (\ker \pi)^\perp$, we have

$$JZ = BZ + CZ, \quad (3.9)$$

where $BZ \in \Gamma(D^\theta \oplus D^\perp)$ and $CZ \in \Gamma(\nu)$.

**Example 3.2.** Let $\pi : \mathbb{R}^8 \to \mathbb{R}^4$ be a Riemannian map defined by

$$\pi(x_1, x_2, ..., x_8) = (x_3, -x_5, \frac{x_1 + x_2}{\sqrt{2}}, x_4).$$

Then the map $\pi$ is a proper quasi-hemi-slant submersion such that

$$D = \text{span}\{\partial x_6, \partial x_7\}, \quad D^\theta = \text{span}\{\partial x_1 - \partial x_2\},$$

with the slant angle $\theta = \frac{\pi}{4}$ and

$$D^\perp = \text{span}\{\partial x_8\}.$$ 

Moreover,

$$(\ker \pi)^\perp = \text{span}\{\partial x_3, -\partial x_5, \partial x_1 + \partial x_2, \partial x_4\}.$$ 

Therefore, using $(3.4)$, $(3.8)$ and $(3.9)$, we get the following.

**Lemma 3.3.** Let $\pi$ be a quasi-hemi-slant submersion from an almost hermitian manifold $(M, g_M, J)$ onto a Riemannian manifold $(N, g_N)$. Then, we have

$$\phi D^\theta = D^\theta, \quad \phi D^\perp = \{0\}, \quad B_\omega D^\theta = D^\theta, \quad B_\omega D^\perp = D^\perp.$$ 

On the other hand, comparing tangential and normal components in $(3.4)$, $(3.9)$ and the fact that $J^2 = -I$, we obtain the following.

**Lemma 3.4.** The endomorphisms $\phi$ and $\omega$, $B$ and $C$ in the tangent bundle of $(M, g_m, J)$ satisfy the following identities:
(i) $\phi^2 + B\omega = -1$, (ii) $\omega\phi + C\omega = 0$, (iii) $B\omega + C^2 = -1$, (iv) $\phi B + BC = 0$, where $I$ is the identity operator on the total space of $\pi$.

We also have the following.

**Lemma 3.5.** Let $\pi$ be a quasi-hemi-slant submersion from an almost Hermitian manifold $(M,g_M,J)$ onto a Riemannian manifold $(N,g_N)$. Then

(i) $(\cos^2 \theta)X = -\phi^2 X$,
(ii) $g_M(\phi X, \phi Y) = \cos^2 \theta g_M(X,Y)$
(iii) $g_M(\omega X, \omega Y) = \sin^2 \theta g_M(X,Y)$
for any $X, Y \in \Gamma(D^\theta)$.

**Proof.** The proof is the same as the one found in [9] \hfill $\Box$

From [1] we can Define the tensors $T$ and $A$ by

$$A_E F = H\nabla_{HE} VF + V\nabla_{HE} HF,$$
$$T_E F = V\nabla_{VE} VF + V\nabla_{VE} HF,$$

for any vector fields $E, F$ on $M$, where $\nabla$ is the Levi-Civita connection of $g_M$. It is easy to see that $T_E$ and $A_E$ are skew-symmetric operators on the tangent bundle of $M$ reversing the vertical and the horizontal distributions. We summarize the properties of the tensor fields $T$ and $A$. Let $U, V$ be vertical and $\xi, \eta$ be horizontal vector fields on $M$, then we have

$$\nabla_U V = T_U V + V\nabla_U V,$$
$$\nabla_U \xi = T_U \xi + H\nabla_U \xi,$$
$$\nabla_U \eta = A_U \xi + \nabla_U \eta,$$
$$\nabla_U \eta = H\nabla_U \xi + A_U \eta,$$

where $H\nabla_U \xi = A_U V$, if $\xi$ is basic. It is not difficult to observe that $T$ acts on the fibers as the second fundamental form, while $A$ acts on the horizontal distribution and measures of the obstruction to the integrability of this distribution. For details on Riemannian submersions, we refer to O'Neill’s paper [1].

Let $(M, g)$ be a Kähler manifold. Then $M$ admits a tensor $J$ of type $(1,1)$ on $M$ such that, for any $X, Y \in \Gamma(TM)$, we have

$$J^2 = -I, \quad g(JX, JY) = g(X, Y), \quad (\nabla_X J)Y = 0,$$

where $g$ is the Riemannian metric and $\nabla$ is the Levi-Civita connection on $M$.

Let $(N, g_N)$ be a Riemannian manifold. We now examine how the Kählerian structure on $M$ effects the tensor fields $T$ and $A$ of a quasi-hemi-slant submersion $\pi : (M, g, J) \rightarrow (N, g_N)$.

**Lemma 3.6.** Let $\pi$ be a quasi-hemi-slant submersion from a Kähler manifold $(M,g_M,J)$ onto a Riemannian manifold $(N,g_N)$. Then

$$V\nabla_U \phi V + T_U \omega V = \phi V\nabla_U V + B T_U V,$$
$$T_U\phi V + H\nabla_U \omega V = \omega V\nabla_U V + C T_U V,$$
$$V\nabla_U \xi + A_U \eta = \phi A_U \eta + B H\nabla_U \xi,$$
$$A_U B \eta + H\nabla_U \xi = \omega A_U \eta + C H\nabla_U \xi,$$
$$V\nabla_U B \xi + T_U \xi = T_U \xi + B H\nabla_U \xi,$$
$$T_U B \xi + H\nabla_U \xi = \omega T_U \xi + C H\nabla_U \xi,$$

where $U, V \in \Gamma(ker \pi)$ and $\xi, \eta \in (ker \pi)^\perp$. 

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Proof. The proof follows from a straightforward calculation using (3.5), (3.9), (3.11) - (3.14) and (3.15).

Now, we define (see [13] for more details)

\[(\nabla_X \phi) Y := \nabla_X \phi Y - \phi \nabla_X Y, \quad (3.22)\]
\[(\nabla_X \omega) Y := \mathcal{H} \nabla_X \omega Y - \omega \nabla_X Y, \quad (3.23)\]
\[(\nabla_Z \mathcal{C}) W := \mathcal{H} \nabla_Z \mathcal{C} W - \mathcal{C} \mathcal{H} \nabla_Z W, \quad (3.24)\]
\[(\nabla_Z \mathcal{B}) W := \mathcal{V} \nabla_Z \mathcal{B} W - \mathcal{B} \mathcal{V} \nabla_Z W, \quad (3.25)\]

for any \(X, Y \in \Gamma(\ker \pi)\) and \(Z, W \in \Gamma((\ker \pi)^{\perp})\). Thus, we have the following.

Lemma 3.7. Let \(\pi\) be a quasi-hemi-slant submersion from a Kähler manifold \((M, g_M, J)\) onto a Riemannian manifold \((N, g_N)\). Then we have

(i) \((\nabla_X \phi) Y = \mathcal{B} \nabla_X Y - \mathcal{T} \nabla_X \omega Y,\)
(ii) \((\nabla_X \omega) Y = \mathcal{C} \nabla_X Y - \mathcal{T} \nabla_X \phi Y,\)
(iii) \((\nabla_Z \mathcal{C}) W = \omega \mathcal{A} Z \mathcal{W} - \mathcal{A} \omega \mathcal{Z} \mathcal{W},\)
(iv) \((\nabla_Z \mathcal{B}) W = \phi \mathcal{A} Z \mathcal{W} - \mathcal{A} \phi \mathcal{Z} \mathcal{W},\)

for any vectors \(X, Y \in \Gamma(\ker \pi)\) and \(Z, W \in \Gamma((\ker \pi)^{\perp})\).

If the tensors \(\phi\) and \(\omega\) are parallel with respect to the linear connection \(\nabla\) on \(M\), respectively, then

\[\mathcal{B} \nabla_X Y = \mathcal{T} \nabla_X \omega Y,\]
and
\[\mathcal{C} \nabla_X Y = \mathcal{T} \nabla_X \phi Y,\]

for any \(X, Y \in \Gamma(TM)\).

4 Integrability of Distributions and Decomposition Theorems

After the introduction of a new Riemannian submersion, it is necessary to study some of the geometric properties of the distributions. Here we examine the integrability conditions for the slant distribution and the anti-invariant distribution.

Theorem 4.1. Let \(\pi\) be a proper quasi-hemi-slant submersion from a Kählerian manifold \((M, g, J)\) onto a Riemannian manifold \((N, g_N)\). Then, the slant distribution \(\mathcal{D}_\theta\) is integrable if and only if

\[g(\mathcal{H} \nabla_Z \omega W - \mathcal{H} \nabla_W \omega Z, JRX) = g(\mathcal{T} \nabla_Z \phi W - \mathcal{T} \nabla_W \phi Z, X)\]
\[- g(\mathcal{V} \nabla_Z \omega W - \mathcal{V} \nabla_W \omega Z, JPX),\]

(4.1)

for any \(Z, W \in \Gamma(\mathcal{D}_\theta)\) and \(X \in \Gamma(\mathcal{D} \oplus \mathcal{D}^\perp)\).

Proof. For any \(Z, W \in \Gamma(\mathcal{D}_\theta)\) and \(X = P X + RX \in \Gamma(\mathcal{D} \oplus \mathcal{D}^\perp)\), using (3.15) and (3.5), we have

\[g([Z, W], X) = g(\nabla_Z \omega W, JX) - g(\nabla_Z J\phi W, X) - g(\nabla_W \omega Z, JX)\]
\[+ g(\nabla_W J\phi Z, X).\]

(4.2)
Then from (3.12), (3.5) and (3.5), we obtain
\[
g([Z, W], X) = g(\mathcal{H}\nabla_Z\omega W - \mathcal{H}\nabla_W\omega Z, JX) + \cos^2 \theta g([Z, W], X) \\
- g(T_Z\omega\phi W - T_W\omega\phi Z, X) + g(V\nabla_Z\omega W - V\nabla_W\omega Z, JX),
\]
which leads to
\[
sin^2 \theta g([Z, W], X) = g(\mathcal{H}\nabla_Z\omega W - \mathcal{H}\nabla_W\omega Z, JRX) \\
- g(T_Z\omega\phi W - T_W\omega\phi Z, X) + g(V\nabla_Z\omega W - V\nabla_W\omega Z, J\rho X),
\]
which completes proof.

From Theorem 4.1, we have the following sufficient conditions for the slant distribution to be integrable

**Corollary 4.2.** Let \( \pi \) be a proper quasi-hemi-slant submersion from a Kählerian manifold \((M, g, J)\) onto a Riemannian manifold \((N, g_N)\). If
\[
\mathcal{H}\nabla_Z\omega W - \mathcal{H}\nabla_W\omega Z \in \omega D^\theta \oplus \nu, \quad T_Z\omega\phi W - T_W\omega\phi Z \in D^\theta,
\]
and
\[
V(\nabla_Z\omega W - \nabla_W\omega Z) \in D^\perp \oplus D^\theta,
\]
for any \( Z, W \in \Gamma(D^\theta) \), then the slant distribution \( D^\theta \) is integrable.

**Theorem 4.3.** Let \( \pi \) be a quasi-hemi-slant submersion from a Kählerian manifold \((M, g, J)\) onto a Riemannian manifold \((N, g_N)\). Then, the anti-invariant distribution \( D^\perp \) is always integrable.

**Proof.** The proof is similar to the one given in [16] for the hemi-slant case.

Now, for any \( Z, W \in \Gamma(D) \) and \( X = QA + RX \in \Gamma(D^\theta \oplus D^\perp) \), using (3.5) and (3.11), we have
\[
g([Z, W], X) = g(\nabla_Z\phi W, JX) - g(\nabla_W\phi Z, JX) \\
= g(T_Z\phi W - T_W\phi Z, JX) + g(V(\nabla_Z\phi W - \nabla_W\phi Z), \phi QX).
\]
Therefore, we have the following.

**Theorem 4.4.** Let \( \pi \) be a proper quasi-hemi-slant submersion from a Kählerian manifold \((M, g, J)\) onto a Riemannian manifold \((N, g_N)\). Then, the invariant distribution \( D \) is integrable if and only if
\[
g(V(\nabla_Z\phi W - \nabla_W\phi Z), \phi QX) = g(T_W\phi Z - T_Z\phi W, JX),
\]
for any \( Z, W \in \Gamma(D) \) and \( X \in \Gamma(D^\theta \oplus D^\perp) \).

We now investigate the geometry of leaves of distributions. Let \( \pi \) be a proper quasi-hemi-slant submersion from a Kählerian manifold \((M, g, J)\) onto a Riemannian manifold \((N, g_N)\). Using the fact that \( \phi D^\perp = \{0\} \), from (3.16), we have
\[
T_X Y = \phi(\nabla_X Y) + BT_Y X,
\]
for any \( X, Y \in D^\perp \). Interchanging \( X \) and \( Y \) in (4.7) and then subtracting it from (4.7), one has
\[
T_X \omega Y - T_Y \omega X = \phi[X, Y].
\]
By Theorem 4.3, we get \( \phi[X, Y] = 0 \) and since \( \omega X = JX \), therefore
\[
T_X \omega Y = T_Y \omega X.
\]
The following theorems give necessary and sufficient conditions for the horizontal distribution and the vertical distribution to be totally geodesic.
Theorem 4.5. Let \( \pi \) be a proper quasi-hemi-slant submersion from a Kählerian manifold \((M, g, J)\) onto a Riemannian manifold \((N, g_N)\). Then the horizontal distribution \((\text{ker} \pi^\perp)\) defines a totally geodesic foliation on \(M\) if and only if
\[
g(A_\xi, PV + \cos^2 \theta QV) = -g(HV_\xi, \omega \phi PV + \omega \phi QV) - g(A_\xi B\eta + HV_\xi C\eta, \omega V),
\]
for any \(\xi, \eta \in \Gamma((\text{ker} \pi^\perp))\) and \(V \in \Gamma(\text{ker} \pi)\).

Proof. From (3.4), (3.5) and using the fact that \(M\) is Kähler, we have
\[
g(\nabla_\xi V, \eta) = g(\nabla_\xi J\eta, PV) + g(\nabla_\xi J\eta, QV) + g(\nabla_\xi J\eta, JRV)
\]
\[
= g(\nabla_\xi J\eta, PV) + g(\nabla_\xi J\eta, QV) + g(\nabla_\xi J\eta, JQV) + g(\nabla_\xi J\eta, JRV).
\]
Again using Kähler character of \(M\), (3.5) and (3.11) - (3.14), we get
\[
g(\nabla_\xi V, \eta) = g(\nabla_\xi J\eta, PV) + g(\nabla_\xi J\eta, \omega PV) + g(\nabla_\xi J\eta, JQV) + g(\nabla_\xi J\eta, JRV)
\]
\[
= g(\nabla_\xi J\eta, PV) + g(\nabla_\xi J\eta, JQV) + g(\nabla_\xi J\eta, \omega PV) + g(\nabla_\xi J\eta, JQV) + g(\nabla_\xi J\eta, JRV),
\]
Now, since \(\omega V = \omega PV + \omega QV + JRV\) and \(\omega PV = 0\), one obtains
\[
g(\nabla_\xi V, \eta) = g(\nabla_\xi J\eta, PV) - g(HV_\xi, \omega \phi PV + \omega \phi QV)
\]
\[
= g(A_\xi B\eta + HV_\xi C\eta, V),
\]
which completes the proof.

Theorem 4.6. Let \( \pi \) be a proper quasi-hemi-slant submersion from a Kählerian manifold \((M, g, J)\) onto a Riemannian manifold \((N, g_N)\). Then the vertical distribution \(\text{ker} \pi\) defines a totally geodesic foliation on \(M\) if and only if
\[
g(\nabla_U PV + \cos^2 \theta QV, \xi) = g(HV_\xi, \omega \phi PV + HV_U \omega \phi QV, \xi)
\]
\[
= g(\nabla_U \omega V, B\xi) - g(HV_U \omega V, C\xi),
\]
for any \(U, V \in \Gamma(\text{ker} \pi)\) and \(\xi \in \Gamma((\text{ker} \pi^\perp))\).

Proof. For any \(U, V \in \Gamma(\text{ker} \pi)\) and \(\xi \in \Gamma((\text{ker} \pi^\perp))\) and using the fact that \(M\) is Kähler, we have
\[
g(\nabla_U V, \xi) = g(HV_U, \omega \phi PV, \xi) + g(\nabla_U \omega PV, \xi) + g(\nabla_U J\xi, \xi)
\]
\[
= g(\nabla_U \omega PV, J\xi) + g(\nabla_U \omega PV, J\xi) + g(\nabla_U \omega QV, J\xi) + g(\nabla_U J\xi, J\xi)
\]
\[
= g(\nabla_U J\xi, J\xi) + g(\nabla_U \omega PV, J\xi) + g(\nabla_U \omega QV, J\xi) + g(\nabla_U J\xi, J\xi).
\]
Using again \(M\) Kähler, one gets
\[
g(\nabla_U V, \xi) = -g(\nabla_U J\phi PV, \xi) + g(\nabla_U \omega PV, J\xi) + g(\nabla_U J\xi, J\xi)
\]
\[
= g(\nabla_U PV, \xi) + g(\nabla_U \omega PV, \xi) + \cos^2 \theta g(\nabla_U QV, \xi) - g(HV_U \omega \phi PV, \xi)
\]
\[
= g(HV_U \omega \phi QV, \xi) + g(\nabla_U \omega PV, J\xi) + g(\nabla_U \omega QV, J\xi) + g(\nabla_U J\xi, J\xi).
\]
Now, since \(\omega V = \omega PV + \omega QV + JRV, \omega PV = 0\) and using (3.11) - (3.14), we have
\[
g(\nabla_U V, \xi) = g(\nabla_U PV + \cos^2 \theta QV, \xi) - g(HV_U \omega \phi PV + HV_U \omega \phi QV, \xi)
\]
\[
= g(\nabla_U \omega V, B\xi) + g(HV_U \omega V, C\xi),
\]
which completes the proof.
The conditions (4.9) and (4.10) in Theorems 4.5 and 4.6, respectively, generalize the ones found in [16]. From Theorems 4.5 and 4.6, we also have the following decomposition result.

**Theorem 4.7.** Let \( \pi \) be a proper quasi-hemi-slant submersion from a Kählerian manifold \((M, g, J)\) onto a Riemannian manifold \((N, g_N)\). Then the total space is a locally product manifold of the form \( M_{ker \pi^*} \times M_{(ker \pi^*)^+} \), where \( M_{ker \pi^*} \) and \( M_{(ker \pi^*)^+} \) are leaves of \( \ker \pi^* \) and \((\ker \pi^*)^+\), respectively, if and only if (4.9) and (4.10) are satisfied.

We now investigate the geometry of leaves of invariant, anti-invariant distribution and slant distribution.

**Theorem 4.8.** Let \( \pi \) be a proper quasi-hemi-slant submersion from a Kählerian manifold \((M, g, J)\) onto a Riemannian manifold \((N, g_N)\). Then, the anti-invariant distribution \( D^\perp \) defines a totally geodesic foliation if and only if

\[
g(H\nabla_X \omega Y, \omega Z) = g(T_X Y, \omega \phi PZ + \omega \phi QZ),
\]
and

\[
g(\omega T_X JY, \xi) = g(H\nabla_X JY, \xi),
\]

for any \( X, Y \in \Gamma(D^\perp), \ Z \in \Gamma(D \oplus D^\phi) \) and \( \xi \in \Gamma((\ker \pi^*)^+) \).

**Proof.** For any \( X, Y \in \Gamma(D^\perp) \) and \( Z = PZ + QZ \in \Gamma(D \oplus D^\phi) \) and using (3.4) and the fact that \( M \) is Kähler, we have

\[
g(\nabla_X Y, Z) = g(\nabla_X JY, JZ) = g(\nabla_X JY, \phi Z) + g(\nabla_X JY, \omega Z)
\]
\[= g(\nabla_X Y, PZ) + g(\nabla_X Y, B\omega PZ) - g(\nabla_X Y, \omega \phi PZ) + \cos^2 \theta g(\nabla_X Y, QZ)
\]
\[= g(\nabla_X Y, \omega \phi QZ) + g(\nabla_X JY, \omega Z).
\]

(4.14)

The left hand-side of the equation (4.14) gives

\[
g(\nabla_X Y, Z) = g(\nabla_X Y, PZ) + g(\nabla_X Y, QZ)
\]
and using (3.5) and (3.12), the equation (4.14) leads to

\[
g(\nabla_X Y, \sin^2 \theta QZ - B\omega PZ) = -g(T_X Y, \omega \phi PZ + \omega \phi QZ)
\]
\[+ g(H\nabla_X \omega Y, \omega Z),
\]

(4.15)

which gives (4.12). Now for any \( X, Y \in \Gamma(D^\perp) \) and \( \xi \in \Gamma((\ker \pi^*)^+) \), and using (3.4) and (3.13), we have

\[
g(\nabla_X Y, \xi) = g(\nabla_X JY, J\xi) = g(T_X JY, B\xi) + g(H\nabla_X JY, C\xi),
\]

(4.16)

which yields (4.13).

**Theorem 4.9.** Let \( \pi \) be a proper quasi-hemi-slant submersion from a Kählerian manifold \((M, g, J)\) onto a Riemannian manifold \((N, g_N)\). Then, the slant distribution \( D^\theta \) defines a totally geodesic foliation if and only if

\[
g(H\nabla_Z \omega W, JRX) = g(T_Z \omega \phi W, X) - g(T_Z \omega W, \phi P X),
\]

(4.17)

and

\[
g(H\nabla_Z \omega \phi W, \xi) - g(H\nabla_Z \omega W, C\xi) = g(T_Z \omega W, B\xi),
\]

(4.18)

for any \( Z, W \in \Gamma(D^\theta) \), \( X \in \Gamma(D \oplus D^\perp) \) and \( \xi \in \Gamma((\ker \pi^*)^+) \).

**Proof.** For any \( Z, W \in \Gamma(D^\theta) \) and \( X = P X + RX \in \Gamma(D \oplus D^\perp) \) and using (3.4) and the fact that \( M \) is Kähler, we have

\[
g(\nabla_Z W, X) = -g(\nabla_Z J\phi W, X) + g(\nabla_Z \omega W, JX).
\]
Then using (3.4) and (3.12), and the fact that $\omega^P X = 0$, one has,
\[
g(\nabla_Z W, X) = \cos^2 \theta g(\nabla_Z W, X) - g(\nabla_Z \omega^P W, X) + g(\nabla_Z \omega W, \phi^P X) + g(\nabla_Z \omega W, JR X),
\]
which leads to
\[
\sin^2 \theta g(\nabla_Z W, X) = -g(\nabla_Z \omega^P W, X) + g(\nabla_Z \omega W, \phi^P X) + g(\nabla_Z \omega W, JR X).
\]

Similarly, we get
\[
g(\nabla_Z W, \xi) = \cos^2 \theta g(\nabla_Z W, \xi) - g(\nabla_Z \omega^P W, \xi) + g(\nabla_Z \omega W, \phi^P \xi) + g(\nabla_Z \omega W, C\xi).
\]
That is,
\[
\sin^2 \theta g(\nabla_Z W, \xi) = -g(\nabla_Z \omega^P W, \xi) + g(\nabla_Z \omega W, \phi^P \xi) + g(\nabla_Z \omega W, C\xi).
\]

Thus, from (4.19) and (4.20), we have the assertions.

**Theorem 4.10.** Let $\pi$ be a proper quasi-hemi-slant submersion from a Kählerian manifold $(M, g, J)$ onto a Riemannian manifold $(N, g_N)$. Then, the slant distribution $D$ defines a totally geodesic foliation if and only if
\[
\begin{align*}
&g(\nabla_X \phi Y, \phi Q Z) = -g(T_X J Y, J Z), \\
&g(\nabla_X \phi Y, B \xi) = -g(T_X J Y, J \xi),
\end{align*}
\]
for any $X, Y \in \Gamma(D)$, $Z \in \Gamma(D^\theta \oplus D^+) \text{ and } \xi \in \Gamma((\ker \pi_\ast)^+)$. 

**Proof.** For any $X, Y \in \Gamma(D)$, $Z = Q Z + R Z \in \Gamma(D^\theta \oplus D^+)$, using $\omega Y = 0$, one has
\[
g(\nabla_X Y, Z) = g(T_X \phi Y, J Z) + g(\nabla_X \phi Y, \phi Q Z).
\]
Now, for any $X, Y \in \Gamma(D)$ and $\xi \in \Gamma((\ker \pi_\ast)^+),
\[
g(\nabla_X Y, \xi) = g(\nabla_X \phi Y, B \xi) + g(T_X \phi Y, C \xi),
\]
which completes the proof. 

From Theorems 4.8, 4.9 and 4.11, we have the following decomposition theorem for the fibers.

**Theorem 4.11.** Let $\pi$ be a proper quasi-hemi-slant submersion from a Kählerian manifold $(M, g, J)$ onto a Riemannian manifold $(N, g_N)$. Then, the fiber of $\pi$ is a local product Riemannian manifold of the form $M_D \times M_{D^\theta} \times M_{D^+}$, where $M_D$, $M_{D^\theta}$ and $M_{D^+}$ are leaves of $D$, $D^\theta$ and $D^+$, respectively, if and only if the conditions (4.12), (4.13), (4.21), (4.22) hold.

We now obtain a necessary and sufficient conditions for a quasi-hemi-slant submersion to be totally geodesic.

For $U, V \in \ker \pi_\ast$ and $\xi \in \Gamma((\ker \pi_\ast)^+)$, we have
\[
\begin{align*}
g(\nabla_V U, \xi) &= g(\nabla_V J V, J \xi) = g(\nabla_V J PV, J \xi) + g(\nabla_V J Q V, J \xi) + g(\nabla_V J R V, J \xi) \\
&= g(\nabla_V J PV + T_V \omega^P Q V + T_V J R V, B \xi) + g(\cos^2 \theta \nabla_V Q V - H \nabla_V \omega^P Q V, \xi) \\
&\quad + g(T_V J PV + H \nabla_V \omega^P Q V + H \nabla_V J R V, \xi).
\end{align*}
\]
Also, for any $\eta \in \Gamma((\ker \pi_\ast)^+),
\[
\begin{align*}
g(\nabla_V U, \xi) &= g(\nabla_V J V, J \xi) = g(\nabla_V J PV, J \xi) + g(\nabla_V J Q V, J \xi) + g(\nabla_V J R V, J \xi) \\
&= g(A_\eta \omega^P Q V + V \nabla_V J PV + A_\eta J R V, B \xi) + g(\cos^2 \theta \nabla_V Q V - H \nabla_V \omega^P Q V, \xi) \\
&\quad + g(A_\eta J PV + H \nabla_V \omega^P Q V + H \nabla_V J R V, \xi).
\end{align*}
\]
Therefore, we have the following.
Theorem 4.12. Let \( \pi \) be a quasi-hemi-slant submersion from a Kählerian manifold \((M, g, J)\) onto a Riemannian manifold \((N, g_N)\). Then, \( \pi \) is a totally geodesic map if and only if

\[
g(\mathcal{H}_U \omega \phi QV - \cos^2 \theta \nabla_U QV, \xi) = g(\nabla_U J PV + \mathcal{H}_U \omega QV + \mathcal{H}_U J RV, B\xi) + g(\mathcal{H}_U J PV + \mathcal{H}_U \omega QV + \mathcal{H}_U J RV, C\xi).
\] (4.23)

and

\[
g(\mathcal{H}_U \omega \phi QV - \cos^2 \theta \nabla_U QV, \xi) = g(A\omega QV + \mathcal{H}_U \omega QV + \mathcal{H}_U J RV, B\xi) + g(A\omega QV + \mathcal{H}_U \omega QV + \mathcal{H}_U J RV, C\xi).
\]

Next, we study the geometry of quasi-hemi-slant submersions with umbilical fibers. We first recall a fiber of a Riemannian submersion \( \pi \) is called totally umbilical if

\[
\mathcal{T}_U V = g(U, V) H,
\] (4.24)

for any \( U, V \in \Gamma(\ker \pi) \), where \( H \) is the mean curvature vector field of the fiber in \( M \). The fiber is called minimal, if \( H = 0 \), identically (see [4] for more details).

We now give a characterization theorem for the proper quasi-hemi-slant submersion with totally umbilical fibers.

Theorem 4.13. Let \( \pi \) be a proper quasi-hemi-slant submersion with totally umbilical fibers from a Kählerian manifold \((M, g, J)\) onto a Riemannian manifold \((N, g_N)\). Then either the invariant and anti-invariant distributions \( \mathcal{D} \) and \( \mathcal{D}^\perp \) are 2 and 1-dimensional subbundles, respectively, or the mean curvature vector field \( H \) of any fiber \( \pi^{-1}(y) \), \( y \in N \) is perpendicular to \( JD^\perp \). Moreover, if \( \phi \) is parallel, then \( H \in \nu \). Furthermore, if \( \omega \) is parallel, then \( T \equiv 0 \).

Proof. The proof follows from the straightforward calculation. \( \square \)

5 Conclusions

Let \( M \) be a 2m-dimensional almost Hermitian manifold with \( g_M \) a Riemannian metric on \( M \) and almost complex structure \( J \), and \( N \) be a Riemannian manifold with Riemannian metric \( g_N \). Then there is a Riemannian submersion \( \pi : (M, g_M, J) \to (N, g_N) \) such that its vertical distribution \( \ker \pi \) admits three orthogonal distributions \( \mathcal{D} \), \( \mathcal{D}^\phi \) and \( \mathcal{D}^\perp \) which are invariant, slant and anti-invariant respectively.

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Competing Interests

Authors have declared that no competing interests exist.
References


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