The Reciprocal Generalized Inverse Gaussian Frailty with Application in Life Annuity Business

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Authors’ contributions

This work was carried out in collaboration among all authors. Author WO designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Authors PW and JO managed the analyses of the study. Author RT managed the literature searches. All authors read and approved the final manuscript.

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Abstract

Aims: As shown in literature, several authors have adopted various individual frailty mixing distributions as a way of dealing with possible heterogeneity due to unobserved covariates in a group of insurers. This research contribution is to generalize the frailty mixing distribution to nest other classes of frailty distributions not in literature and apply the proposed distributions in valuation of life annuity business.

Methodology: A simulation study is done to assess the performance of the aforementioned models. The baseline parameters is estimated using Bayesian Inference and a better model is suggested for valuation of life annuity business.

Results: As a result of generalizing the frailty some new classes of frailty distributions are constructed such as; the Reciprocal Inverse Gaussian Frailty, the Inverse Gamma Frailty, the Harmonic Frailty and the Positive Hyperbolic Frailty. From the simulation study, the proposed new frailty models shows that ignoring frailty leads to an underestimation of future residual lifetime since the survival curve shifts to the right when heterogeneity is accounted for. This is consistent with frailty literature. The Reciprocal Inverse Gaussian model closely represents the Association of Kenya Insurers graduated rates with a slight increase in survival due to longevity risk.

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Conclusion: The proposed new frailty models show an increase in the insurer’s expected liability when unobserved heterogeneity is accounted for. This is consistent with frailty literature and thus can be applied to avoid underestimating the insurer’s liability in the context of life annuity business. The RIG model as proposed in estimating future liability by directly adjusting the AKI mortality rates shows an increase in longevity risk. The extent of heterogeneity of the insured group determines the level of risk. The RIG frailties should be considered for multivariate cases where the insureds are clustered in groups.

Keywords: Frailty model; generalized inverse Gaussian distribution; reciprocal inverse Gaussian distribution; harmonic distribution; positive hyperbolic distribution; Bayesian inference; life annuity insurance.

1 Introduction

Frailty modeling is based on mixture distributions where the population hazard is considered a mixture of measurable (e.g. health status) and unmeasured (e.g. congenital personal characteristics) risk factors affecting mortality. The frailty model is an extension of the Cox PH model. It is a random effects model which has a multiplicative effect on the hazard rate that adds additional risks based on each individual’s information. The term “frailty” was introduced by Vaupel et al. [1] in a seminal paper on individual survival models. They discuss the impact of heterogeneity in individual mortality. Their findings showed that standard life-table methods overestimates current life expectancy and potential gains in life expectancy from health and safety interventions, while underestimating rates of individual aging, past progress in reducing mortality.

Standard life tables assume that the population under study is homogeneous. This means that all individuals in that study are subject under the same risk at a given age. Basic observation of medical statistics shows that individuals differ greatly see Vaupel et al. [1].

Frailty models have been adopted by several authors in insurance, for instance; Shusu et al. [2] applies frailty to quantify the extent of heterogeneity in Australian population mortality on life annuity rates and pension costs the results confirm significant heterogeneity exists. Ramona and Sherris [3] have used frailty model to quantify the impact of heterogeneity due to underwriting factors and frailty on annuity values the results showed that heterogeneity remains after underwriting and that frailty significantly impacts the fair value of both standard and underwritten annuities.

Annamaria and Pitacco [4] suggests adopting a frailty model for risk classification for life annuity portfolios. In particular, they identify risk groups within a population by assigning specific ranges of values to the frailty within each group. Pitacco E. [5] applies frailty modelling to analyze the impact of frailty, in its various interpretations, on the results of cash flows, profits, etc of life insurance and life annuity portfolios and related risk profiles.

Avanzi et al. [6] bootstrap data on Canadian pensioners’ mortality to study the characteristics of its implied heterogeneity they find strong support for the Gamma frailty model. Nadine G. et al. [7] utilizes the frailty model to reflect mortality heterogeneity in optimal risk classification for substandard annuities. Their findings indicate that extended frailty risk classification can enhance the insurer’s profitability.

Eriksson F and Scheike T [8] applies the additive Gamma frailty models to competing risks in related individuals.
2 Methodology

The multiplicative approach:

\[ h(t \mid u) = uh_0(t) \]

\( h_0(t) \) is the ‘standard hazard function’ corresponding to a ‘standard individual’, conventionally those with frailty \( u = 1 \). The non-negative quantity \( u \) encompasses all other factors affecting mortality other than age which acts in a multiplicative manner. Individuals with \( u > 1 \) experience a force of mortality that is proportionally higher at all ages. Individuals with \( u < 1 \) experience proportionally lower mortality rates.

Frailty models without observed covariates:

This model is used when only survival data is available for the analysis, or when additional information is of no interest. I.e \( h(t \mid u) = uh_0(t) \)

This model is non-identifiable from survival data, since different combinations of \( h_0(t) \) and frailty distributions may produce the same marginal hazard rate \( h(t) \). The model becomes identifiable when the parametric structure of \( h_0(t) \) is fixed and \( u \) is assumed to belong to some parametric distribution family.

Univariate frailty models:

This model accounts for heterogeneity due to unobserved risk factors for independent life times in a proportional hazard model. The variability can be split into a part that depends on observable risk factors, and is therefore theoretically predictable, and a part that is theoretically unpredictable, even when all relevant information is known. This model has been used by several authors Rocha [9] and Hougaard [10] to show that these two sources of variability can explain some unexpected results.

2.1 Model construction

Let \( T \) be the future life-time random variable with a continuous distribution. A non-negative random variable \( u \) is called “frailty” if the conditional hazard function given \( U = u \) is given by;

\[ h(t \mid u) = uh_0(t); t > 0 \]

where \( h_0(t) \) is called the baseline age-specific hazard function for a “standard” individual. The “population” hazard corresponding to a randomly selected individual that is actually observed is given by;

\[ E[h(t \mid u)] = \int uh_0(t)f(u)du = \bar{u} \cdot h_0(t) \]

which increases less rapidly than for individuals. This is because the population becomes populated by more and more robust individuals as the frail members fail. The conditional survival function is given by;

\[ s(t \mid u) = e^{-\int_{u}^{t}h(x)dx} = e^{\int_{0}^{t}h_0(x)dx} \]

\[ s(t \mid u) = e^{-\int_{0}^{t}h_0(x)dx}, t > 0 \]

\( H_0(t) = \int_{0}^{t}h_0(x)dx \) is the cumulative baseline hazard.
Comparing the conditional and unconditional survival functions yields $s(t | u) = \{s(t)\}^u$

This shows that an individual with frailty of level of 2 is twice as likely to die compared to a “standard” individual.

Since the frailty “u” is unobserved and considered random it is integrated out and thus the population is considered a mixture over “u”. The univariate marginal survival function is;

$$s(t) = \int_0^\infty s(t | u)f(u)du = E[s(t | u)] = E[e^{-ut}H_x(t)] = L_u(H_0(t)); t > 0$$  \hspace{1cm} (1)

The contribution is to construct a generalized frailty mixing distribution to nest other individual classes of frailty distributions.

2.1.1 Baseline hazard distribution

The baseline hazard used in this study is the Exponential Distribution.

The Exponential Baseline Distribution.

The density function, survival function and hazard function are;

$$f(t) = \lambda e^{-\lambda t}; t, \lambda > 0$$

$$s(t) = \int_0^\infty \lambda e^{-\lambda u} du = e^{-\lambda t}$$

$$h(t) = \frac{f(t)}{s(t)} = \lambda$$

$$H(t) = \int_0^t h(u)du = \int_0^t \lambda du = \lambda t$$

2.1.2 The generalized inverse Gaussian frailty

The Generalized Inverse Gaussian (GIG) distribution can be constructed under various parametizations. For instance, considering The Sichel’s [11] parametization

$$\omega = \sqrt{\phi \theta} ; \quad \phi, \theta > 0$$

$$K_v(\omega) = \frac{1}{2} \int_0^\infty z^{v-1} e^{-\frac{\omega}{2}(z+\frac{1}{z})} dz$$

where $K_v(\omega)$ is a Bessel function of the third kind with order $w$ and index $v$.

Using the transformation: $z = \sqrt{\frac{\omega}{\theta} x}, dz = \sqrt{\frac{\omega}{\theta}} dx$
\[ K_v(\sqrt{\varphi \theta}) = \frac{1}{2} \int_0^\infty \left( \sqrt{\frac{\varphi}{\theta}} \right)^v x^{v-1} e^{-\frac{\sqrt{\varphi \theta}}{2} \left( x \sqrt{\frac{\varphi}{\theta}} + \frac{1}{x} \right)} dx \]

\[ 1 = \frac{\int_0^\infty \left( \sqrt{\frac{\varphi}{\theta}} \right)^v x^{v-1} e^{-\frac{1}{2} \left( \psi_m + \frac{\theta}{x} \right)} dx}{2K_v(\sqrt{\varphi \theta})} \]

\[ f(x) = \frac{\left( \sqrt{\frac{\varphi}{\theta}} \right)^v x^{v-1} e^{-\frac{1}{2} \left( \psi_m + \frac{\theta}{x} \right)}}{2K_v(\sqrt{\varphi \theta})}; x > 0, \varphi > 0, \theta > 0, -\infty < \nu < \infty \] (2)

Is the probability density function of a GIG(\nu, \varphi, \theta)

The Laplace transform is

\[ L_u(s) = E[e^{-su}] = \frac{\int_0^\infty u^{v-1} \left( \sqrt{\frac{\varphi}{\theta}} \right)^v e^{-\frac{1}{2} \left( \psi_m + 2 \psi_s + \frac{\theta}{u} \right)} du}{2K_v(\sqrt{\varphi \theta})} \]

\[ L_u(s) = \frac{\left( \sqrt{\frac{\varphi}{\theta}} \right)^v 2K_v(\sqrt{\varphi+2s})}{\left( \sqrt{\frac{\varphi+2s}{\theta}} \right)^v 2K_v(\sqrt{\varphi \theta})} = \frac{\left( \sqrt{\frac{\varphi}{\theta}} \right)^v K_v(\sqrt{\varphi+2s})}{K_v(\sqrt{\varphi \theta})} \] (3)

The marginal survival function at time \( t > 0 \)

\[ S(t) = L_u(H_0(t)) \]

\[ S(t) = \frac{\left( \sqrt{\frac{\varphi}{\theta+2(H_0(t))}} \right)^v K_v(\sqrt{\varphi+2H_0(t)})}{K_v(\sqrt{\varphi \theta})} \]

**Special cases**

**Case 1: Inverse Gaussian Distribution (IG):**

Let \( \nu = -\frac{1}{2} \) in equation (3)
\[ L_U(s) = \frac{(\frac{\phi}{\phi+2s})^{\frac{1}{2}} K_{\frac{1}{2}}(\frac{\phi+2s}{\sqrt{2}}) \theta)}{K_{\frac{1}{2}}(\theta \sqrt{\phi \theta})} \]  

\[ L_U(s) = \frac{\left(\frac{\phi}{\phi+2s}\right)^{\frac{1}{2}} \frac{\pi}{\sqrt{2(\phi+2s)\theta}} e^{-\left(\sqrt{\phi+2s}\theta\right)}}{\pi \sqrt{2\sqrt{\phi\theta} e^{-\sqrt{\phi\theta}}}} \]  

\[ L_U(s) = e^{\sqrt{\phi\theta}-\sqrt{(\phi+2s)\theta}} \]  

(4)

Substituting \( \phi = \frac{1}{\beta^2}, \theta = \mu^2 \) we get

\[ L_u(s) = \exp \left\{ -\frac{\mu}{\beta} [(1 + 2\beta^2s)^{1/2} - 1] \right\} \]

Mean = \(-L'_u(0) = \mu\beta\)

Variance = \(L''_u(0) - (L'_u(0))^2 = \mu\beta^3\)

For identifiability reasons the mean is normalized to one. i.e. \( \mu\beta = 1; \mu = \frac{1}{\beta} \) thus the variance \( \delta^2 = \beta^2 \)

The Laplace transform is therefore

\[ L_U(s) = \exp \left[ \frac{1-(1+2s\delta^2)^{1/2}}{\delta^2} \right] \]

The marginal survival function at time \( t > 0 \)

\[ S(t) = L_u(H_0(t)) \]

\[ S(t) = \exp \left[ \frac{1-(1+2(H_0(t))\delta^2)^{1/2}}{\delta^2} \right] \]

Case 2: Reciprocal Inverse Gaussian Distribution (RIG)

Let: \( v = \frac{1}{2} \) in equation (3)
\[ L_U(s) = \frac{\sqrt{\frac{\phi}{\phi+2s}}^{\frac{1}{2}} K_1(\sqrt{(\phi+2s)\theta})}{K_1(\sqrt{\phi\theta})} \]

\[ L_U(s) = -\left(\frac{\phi}{\phi+2s}\right)^{\frac{1}{2}} \sqrt{\frac{\pi}{2(\sqrt{(\phi+2s)\theta})}} e^{-(\sqrt{(\phi+2s)\theta})} \]

\[ L_U(s) = \left(\frac{\phi}{2s+\phi}\right)^{\frac{1}{2}} e^{\sqrt{\phi\theta}-\sqrt{(\phi+2s)\theta}} \]

\[ L_U(s) = (1 + \frac{2s}{\phi})^{-\frac{1}{2}} e^{\sqrt{\phi\mu(1-\sqrt{\frac{2s}{\phi}})}} \]

Substituting \( \phi = \frac{1}{\beta^2}, \theta = \mu^2 \) we get

\[ L_U(s) = (1 + 2s\beta^2)^{-\frac{1}{2}} e^{\mu(1-\sqrt{1+2s\beta^2})} \]

For identifiability the mean is normalized to one i.e.

\[ E[U] = -L_U'(s) = \{\beta^2(1 + 2s\beta^2)^{-\frac{3}{2}} + \mu\beta(1 + 2s\beta^2)^{-\frac{1}{2}}\} e^{\mu(1-\sqrt{1+2s\beta^2})} \]

\[ E[U] = -L_U'(0) = \beta^2 + \mu\beta \]

\[ \beta^2 + \mu\beta = 1; \mu = \frac{1-\beta^2}{\beta} \]

The Laplace becomes:

\[ L_U(s) = (1 + 2s\beta^2)^{-\frac{1}{2}} e^{\frac{1-\beta^2}{\beta^2(1-\sqrt{1+2s\beta^2})}} \]

The marginal survival function at time \( t > 0 \)
\[ S(t) = L_U(H_0(t)) \]

\[ S(t) = (1 + 2\beta^2 H_0(t))^{-\frac{1}{2}} e^{\frac{1-\beta^2}{\beta^2} \frac{1}{1+2\beta^2 H_0(t)}} \]

**Case 3: Gamma Distribution:**

Let \( \theta = 0, \nu > 0, \varphi > 0 \) in equation (3)

\[ L_U(s) = \left( \frac{\varphi}{\varphi + 2s} \right)^{s/2} = \left( \frac{2s}{\varphi} \right)^{s/2} \]

Substituting \( \varphi = 2b, \nu = 2p \) we get

\[ L_U(s) = (1 + \frac{s}{b})^{-p} \]

\[ \text{Var}(z) = (L''(0) - (L'(0))^2) = -p(-p - 1)^p \left( \frac{s}{b} \right)^{-p-2} \left( \frac{1}{b} \right)^2 - \left( \frac{p}{b} \right)^2 = \frac{p}{b^2} \]

For purposes of identifiability assume the distribution of \( U \) has mean normalized to one (\( p=b \)) and variance

\[ \sigma^2 = \frac{1}{b} \]

The Laplace becomes

\[ L(s) = (1 + s\delta^2)^{-1/\delta^2} \]

The marginal survival function at time \( t > 0 \)

\[ S(t) = L_U(H_0(t)) \]

\[ S(t) = \{1 + \delta^2 H_0(t)\}^{-1/\delta^2} \]

**Case 4: The Levy Distribution:**

This is a special case of the Inverse Gaussian distribution.

Let \( \theta > 0, \nu = -\frac{1}{2}, \varphi = 0 \) in equation (4)

\[ L_U(s) = e^{\varphi s - \sqrt{(\varphi + 2s)\varphi}} \]

when \( \varphi = 0 \)

\[ L_U(s) = e^{-\sqrt{2s\theta}} \]
For purposes of identifiability assume the distribution of $U$ has mean normalized to 1. e.g. $E[U] = \theta = 1$

The Laplace becomes

$$L_U(s) = e^{-\sqrt{2}s}$$

The marginal survival function at time $t > 0$

$$S(t) = L_U(H_0(t))$$
$$S(t) = e^{-\sqrt{2}H_0(t)}$$

**Case 5: The Harmonic Distribution**

Let $\theta = an, \nu = 0, \varphi = \frac{a}{n}$ in equation (3)

The Laplace becomes;

$$L_U(s) = \frac{K_\nu(\sqrt{2ans+a^2})}{K_\nu(a)}$$

For purposes of identifiability assume the distribution of $U$ has mean normalized to 1.

$$E[U] = \frac{nK_\nu(1/n)}{K_\nu(a)} = 1$$

The Laplace becomes

$$L_U(s) = \frac{K_\nu(\sqrt{2ans+a^2})}{nK_\nu(1/n)}$$

The marginal survival function at time $t > 0$

$$S(t) = L_\nu(H_0(t))$$
$$S(t) = \frac{K_\nu(\sqrt{2anH_0(t)+a^2})}{nK_\nu(1/n)}$$

**Case 6: The Positive Hyperbolic Distribution**

Let $\theta > 0, \nu = 1, \varphi > 0$ in equation (3)

The Laplace becomes;
\[ L_U(s) = \left( \frac{\phi}{\sqrt{\phi^2 + 2s}} \right) K_1(\sqrt{(\phi^2 + 2s)\theta}) \]

For purposes of identifiability assume the distribution of \( U \) has mean normalized to 1. e.g.

Let \( \theta > 0, \nu = 1, \phi > 0 \) in equation (2)

\[ f(u) = \frac{\phi}{\sqrt{\theta}} \frac{e^{-\frac{1}{2}(\phi^2 + \theta)u}}{2K_1(\sqrt{\phi\theta})} u > 0; \theta > 0; \phi > 0 \]

\[ E[U^r] = \frac{\phi}{\sqrt{\theta}} \frac{1}{2K_1(\sqrt{\phi\theta})} \int_0^\infty u^r e^{-\frac{1}{2}(\phi^2 + \theta)u} du = \left( \frac{\theta}{\phi} \right)^r \frac{K_{1+r}(\sqrt{\phi\theta})}{K_1(\sqrt{\phi\theta})} \]

\[ \therefore E[U] = \left( \frac{\theta}{\phi} \right) \frac{K_1(\sqrt{\phi\theta})}{K_1(\sqrt{\phi\theta})} = 1 \]

The Laplace becomes

\[ L_U(s) = \left( \frac{\phi^2}{\phi^2 + 2s} \right) K_1(\sqrt{(\phi^2 + 2s)\theta}) \]

The marginal survival function at time \( t > 0 \)

\[ S(t) = L_U(H_0(t)) \]

\[ S(t) = \left( \frac{\phi^2}{\phi^2 + 2H(t)} \right) K_1(\sqrt{(\phi^2 + 2H(t))\theta}) \]

3 Results and Discussion

3.1 Simulation study

A simulation study is done to check the performance of the proposed models. A comparison is done on the survival function in the presence of different levels of frailty (heterogeneity) and without frailty
homogeneity). The baseline parameters are estimated via Bayesian analysis using Gibbs Sampler. RCODE is shown in APPENDIX.

CASE 1

Gamma-Exponential Model

The Survival function without frailty: 
\[ S(t) = e^{-\lambda t} \]

The Survival function with frailty: 
\[ S(t) = (1 + \delta^2 \lambda t)^{-1/\beta^2} \]

\( \lambda = 0.05 \) (failure rate), \( \delta^2 = 5, 30, 60 \) (frailty levels) t=55:110 (age)

CASE 2

Inverse Gaussian-Exponential Model

The Survival function without frailty: 
\[ S(t) = e^{-\lambda t} \]

The Survival function with frailty: 
\[ S(t) = \exp\left\{\frac{1-(1+2\delta^2 \lambda t)^{0.5}}{\delta^2}\right\} \]

\( \lambda = 0.05 \) (failure rate), \( \delta^2 = 5, 30, 60 \) (frailty levels) t=55:110 (age)

CASE 3

Reciprocal Inverse Gaussian-Exponential Model

The Survival function without frailty: 
\[ S(t) = e^{-\lambda t} \]

The Survival function with frailty: 
\[ S(t) = (1 + 2\beta^2 \lambda t)^{-0.5} \exp\left\{\frac{1-\beta^2}{\beta^2} * (1 - (1 + 2\beta^2 \lambda t)^{0.5})\right\} \]

\( \lambda = 0.05 \) (failure rate), \( \beta = 0.1 \), t=55:110 (age)

CASE 4

Levy-Exponential Model

The Survival function without frailty: 
\[ S(t) = e^{-\lambda t} \]

The Survival function with frailty: 
\[ S(t) = e^{-(2\lambda t)^{0.5}} \]

CASE 5

Harmonic-Exponential Model

The Survival function without frailty: 
\[ S(t) = e^{-\lambda t} \]

The Survival function with frailty: 
\[ S(t) = \frac{K_n(\sqrt{2an\lambda t + a^2})}{nK_n(1/n)} \]
a=0.05, n=2.5 (shape and scale parameters)

CASE6

Positive Hyperbolic-Exponential Model

The Survival function without frailty: $S(t) = e^{-\lambda t}$

The Survival function with frailty: $S(t) = \left(\frac{\varphi^2}{(\varphi+2\lambda t)^\theta}\right) K_0(\sqrt{(\varphi+2\lambda t)\theta})

\varphi^2 = \left(\sqrt{\theta(\varphi+2\lambda t)^\theta}\right) K_2(\sqrt{\varphi\theta})

\theta=10, \varphi=0.03$ (shape and scale parameters)

Fig. 1. Graphical representation of exponential model
Discussion:

From the simulation study above the proposed new frailty models shows that ignoring frailty leads to an underestimation of future residual lifetime since the survival curve shifts to the right when heterogeneity is accounted for this is consistent with frailty literature. The aforementioned models can therefore be applied in insurance to avoid underestimating the insurer’s liability in the context of life annuity business.

Parameter estimation:

Model parameters are fixed quantitative values that characterize the model believed to reflect the real world. They have to be estimated either by statistical inference from observations or by expert opinion. Since the baseline model assumes the population to be homogeneous, the Association of Kenya Insurer’s (AKI-2010) graduated rates will be considered as the baseline.

3.2 Applications in insurance industry

The pricing of long term insurance, annuity and pension products is largely influenced by the choice of the mortality projection model. Frailty models are used in life insurance to represent heterogeneity in a population due to non-observed risk factors. Heterogeneity due to observable risk factors is addressed at policy issue during the underwriting process to ensure that each contract is assigned premium consistent with the insured risk. Neglecting such factors or use of age and sex as the only rating factors (see Joelle F. [12]) may lead to mispricing of insurance products.

The aims of this exercise are:

- The first aim is to show that when heterogeneity is disregarded the expected residual lifetime is underestimated thus leading to an underestimation of the insurer’s liability.
- Secondly, is to show the relevance of the proposed Reciprocal Inverse Gaussian Frailty mixture to reflect the insurer’s mortality rating.

Assumption:

- The force of mortality $\mu$ is assumed piece-wise constant, taking a common value across each whole year of age $[x, x+1)$ similar assumption found in Dodd E et al. [13]
- The frailty model considered here is one without observed covariates since only survival data is available for analysis.
- Maximum age is 109
- Fixed annual interest rate of 2%

3.3 Data analysis and results

The Data:

3.3.1 The Kenyan life tables 2007-2010

The AKI 2010 table of mortality is based upon data collected by the Association of Kenya Insurers for an investigation into the mortality of assured lives in the Republic of Kenya. The method used was the census method between 2007-2010 inclusive. The AKI 2010 mortality rates will be used as the baseline hazard rate in the study.

Consider two hypothetical insurers i.e. insurer X and Y
Insurer X assumes the population to be heterogeneous with respect to underwriting factors and applies the Life-tables from the Association of Kenya Insurers (AKI 2010) graduated rates.

Insurer Y assumes the population to be heterogeneous with respect to both observable and unobserved risk factors and applies frailty modeling to modify the AKI q-rates.

The Reciprocal Inverse Gaussian-Exponential Model

\[ S(t) = (1 + 2\beta^2 \bar{H}_o(t))^{0.5} \exp\left\{ \frac{\beta^2}{\beta^2} \right\} \exp\left\{ \frac{1 - \beta^2}{\beta^2} \right\} \exp\left\{ 1 - (1 + 2\beta^2 \bar{H}_o(t))^{0.5} \right\} \]

Where \( \bar{H}_o(t) \)– AKI 2010 q-rates

**Fig. 2. AKI and frailty survival functions**

**Discussion:**

1. Ignoring heterogeneity due to other factors affecting mortality other than age and sex ie applying the AKI 2010 rates as it is leads to an underestimation of life expectancy as the survival curve shifts to the right when heterogeneity is accounted for.
2. The Reciprocal Inverse Gaussian model closely represents the AKI 2010 graduated rates with a slight increase in survival due to longevity risk.

**3.3.2 Deferred life annuity business**

These are annuities which commence in m (say) years' time, provided that the annuitant is then active. Thus the present value of amount b payable for a future lifetime \( T_{x+t} \)

\[ m \left| \bar{a}_x = \frac{D_{x+m}}{D_x} \right| a_{x+m} \]
Where \( \frac{D_{x+m}}{D_x} \) is a pure endowment factor and \( a_{x+m} \) is an annuity factor at age \( x+m \).

**Fig. 3. AKI and RIG frailty insurers expected liability**

**Discussion:**

1. When heterogeneity is disregarded the expected liability is underestimated.
2. The Reciprocal Inverse Gaussian frailty is a close estimate of the insurer liability with a slight increase due to observed heterogeneity.

**4 Conclusion**

The proposed new frailty models show an increase in the insurers expected liability when unobserved heterogeneity is accounted for. This is consistent with frailty literature and thus can be applied to avoid underestimating the insurer’s liability in the context of life annuity business.

The RIG model as proposed in estimating future liability by directly adjusting the AKI mortality rates shows an increase in longevity risk. The extent of heterogeneity of the insured group determines the level of risk. The RIG frailties should be considered for multivariate cases where the insureds are clustered in groups.

**Competing Interests**

Authors have declared that no competing interests exist.

**References**


APPENDIX 1A

WINBUGS CODE

MODEL<-FUNCTION()
FOR(I IN 1:55){
S[I] ~ DEXP(LAMBDA)
LAMBDA ~ DEXP(0.09)
}
WRITE.MODEL(MODEL,"MODEL.TXT")
INIT<-FUNCTION(){LIST(ALPHA=DEXP(0.9))}
DATA=LIST(S=GIG_TABLE$S.X._AKI.MALE.I.2.)
BUGS=BUGS(DATA=DATA,INITS=INIT,PARAMETERS.TO.SAVE=C("LAMBDA"),MODEL.FILE="MODEL.TXT",BUGS.DIRECTORY="C:/USERS/DELL/DOCUMENTS/R/WINLIBRARY/3.5/R2WINBUGS",N.CHAINS=1,N.ITER=100000,N.BURNIN=100,CODAPKG=TRUE,DEBUG=T)

Node Statistics

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CASE 1 R-CODE:

```r
PAR(MFROW=C(2,2))
AGE=56:1
SURVIVAL_FUNCTION=SEQ(0.1,1/5)
LAMBDA=1.115# WINBUGS ESTIMATE
S1 <- (6+5*LAMBDA*AGE)^(-1/6)
S2 <- (6+30*LAMBDA*AGE)^(-1/30)
S3 <- (6+60*LAMBDA*AGE)^(-1/60)
S4 <- EXP(-LAMBDA*AGE)
PLOT(AGE,SURVIVAL_FUNCTION,TYPE="N",MAIN="GAMMA-EXPONENTIAL MODEL")
LINES(AGE,S1,COL="BLACK")
LINES(AGE,S2,COL="BLUE")
LINES(AGE,S3,COL="GREEN")
LINES(AGE,S4,COL="RED")
```

CASE 2 R-CODE:

```r
S1 <- EXP((1-(1+2*5*LAMBDA*AGE)^0.5)/5)
S2 <- EXP((1-(1+2*30*LAMBDA*AGE)^0.5)/30)
S3 <- EXP((1-(1+2*60*LAMBDA*AGE)^0.5)/60)
PLOT(AGE,SURVIVAL_FUNCTION,TYPE="N",MAIN="IGAUSSIAN-EXPONENTIAL MODEL")
LINES(AGE,S1,COL="BLACK")
LINES(AGE,S2,COL="BLUE")
LINES(AGE,S3,COL="GREEN")
LINES(AGE,S4,COL="RED")
```

CASE 3 R-CODE:

```r
BETA=0.9
S1<-(1+2*BETA*LAMBDA*AGE)^(-0.5)*EXP((1-BETA)*(1-(1+2*BETA*LAMBDA*AGE)^0.5)/BETA)
PLOT(AGE,SURVIVAL_FUNCTION,TYPE="N",MAIN="R.I.G -EXPONENTIAL MODEL")
LINES(AGE,S1,COL="BLACK")
LINES(AGE,S4,COL="RED")
```

CASE 4 R-CODE:

```r
S1 <- EXP(-SQRT(2*LAMBDA*AGE))
PLOT(AGE,S1,MAIN="LEVY-EXPONENTIAL MODEL",TYPE="L")
LINES(AGE,S4,COL="RED")
```

CASE 5 R-CODE:

```r
PAR(MFROW=C(1,2))
A=0.05
N=2.5
S1 <- BESSELY(SQRT(2*A*N*LAMBDA*AGE+2),0)/N*BESSELY(1/N,1)
PLOT(AGE,S1,YLAB="SURVIVAL FUNCTION",TYPE="O",MAIN="HARMONIC-EXPONENTIAL MODEL")
LINES(AGE,S4,COL="RED")
```
CASE 6 R-CODE:

VARPHI=10
THETA=0.03
SI<-SQRT((VARPHI^2)/(VARPHI+2*LAMBD*A*AGE)*THETA)*(BESSELK(SQRT((2*LAMBD*A*AGE+VARPHI)*THETA),1))/BESSELK(SQRT(THETA*VARPHI),2)
PLOT(AGE,S1,YLAB="SURVIVALFUNCTION",TYPE="O",MAIN="POSITIVE HYPERBOLIC-EXPONENTIAL MODEL")
LINES(AGE,S4,COL="RED")

COMPARISON GRAPHS 1 R-CODE

GIG_TABLE=READ.CSV("C:/USERS/DELL/DESKTOP/PROJECTS/GIG_BOOK1.CSV",HEADER=T)
PLOT(AGE.X,S.X_AKI.MALE.I2.,TYPE="O",MAIN="AKI AND FRAILTY SURVIVAL FUNCTIONS",YLAB="SURVIVAL_FUNCTION",XLAB="AGE")
BETA=0.9
RIG_SX<-((1+2*BETA*BETA*GIG_TABLESHX_AKI)^(-0.5))*EXP((1-BETA*BETA)*(1-
(1+2*BETA*BETA*GIG_TABLESHX_AKI)^(-0.5)))/(BETA*BETA)
LINES(AGE.X,RIG_SX,COL="GREEN",TYPE="O")

COMPARISON GRAPHS 2 R-CODE

PLOT(AGE.X,AX_AKI,TYPE="O",MAIN="AKI AND RIG FRAILTY INSURERS EXPECTED LIABILITY",YLAB="EXPECTED LIABILITY",XLAB="AGE")
lines(Age.x,ax_RIG,col="red")

Table 1. Association of Kenya Insurers Table I=2%

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Table 2. Reciprocal Inverse Gaussian frailty table

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