An Alternative Approach to Solving Cubic and Quartic Polynomial Equations

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Author’s contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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Abstract

Aims: The aim of the research study was to develop a more direct and intuitive approach for the solution of polynomial equations of degree 3 and four.

Study Design: The study employed equivalent polynomial substitution that is more intuitive and direct to formulate than the traditional formulations and one that is easily solvable.

Place and Duration of Study: The study has been undertaken by the author at the university of Eswatini in the period from February to March 2021.

Methodology: Two alternative procedures have been presented for the analytical solution of cubic and quartic equations and demonstrated with worked examples. The solution is derived through a direct procedure without involving intermediate variable substitution.

Results: For cubic equations, the solution provides explicit expression of an equivalent cubic that is formed directly in terms of the original variable x. As such, the formula is intuitive and simple to derive or understand as well as apply. For the quartic equations, the same decomposition form is used as that of the cubic equation using two quadratic polynomials that have symmetric form thus making it easy to develop the solution as well as solve the equations

Conclusion: The alternative formula is easy to formulate and solve and provides a more intuitive basis for understanding and solving polynomial equations.

Keywords: Polynomial equations; cubic equations; quartic equations; mathematics; algebra; roots of equations.

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1 Introduction

The methods for solving polynomial equations in the form of quadratic and cubic equations appeared since the early periods of Babylonian mathematicians around 2000 BC. During the Renaissance period (1450-1630), algebraic solutions to cubic and quartic polynomial equations were successfully established by Italian mathematicians. Del Ferro in 1505 solved the cubic equation that is given in reduced form though his solution was not made public [1]. Nicolo Brescia (known also as Tartaglia) also developed the solution to the cubic equation which he revealed only to Hieronimo Cardan. Cardan eventually published the solution to the cubic equation whereas crediting Del Ferro for the method as the earlier discoverer than Tartaglia.

Following the ground work laid by Cardan and Tartaglia, Francois Vieta (1540-1603) generalized the solution to cubic equations by introducing changes of variable. Joseph Louis Lagrange (1736-181) used combination rather than substitution to solve cubic polynomial equations. Lagrange considered the three roots of cubic equation as forming symmetric functions involving one of the complex conjugate cubic roots of unity, \( w \), as coefficient and the roots of cubic equations \( x_1, x_2 \) and \( x_3 \) as variables. The roots are obtained by solving two combinations of the symmetric functions that are expressed in terms of the coefficients of the original cubic equation. Lagrange’s solution as such implicitly involved discrete Fourier transform before such method was even established. Lagrange’s method also served as precedent to Gallois Theory on conditions for insolvability of polynomials of degree greater than or equal to five [2].

A more intuitive procedure for solving cubic equations has been provided by Mukundan [3]. Abesheck Dak simplified the solution procedure for cubic and quartic by making use of derivative values related to the stationary points [4]. Tiruneh [5] proposed a method of functional evaluation that generalizes simplification of the solution to polynomial equations in terms of functional values and derivatives evaluated at stationary points.

The solution to quartic equations was provided first by Ferari (1522-1565) which involved a resolvent cubic equation in the process [6]. René Descartes (1596-1650) also provided an alternative solution of the quartic equation that reduces the original quartic equation as products of two quadratic polynomials [7]. His method also involved solving a resolvent cubic equation. The occurrence of repeating root implies that the resolvent cubic also has repeating root. However, the repeating root is obtained from the non-repeating part of the root of the resolvent cubic [8].

Leonhard Euler (1707-1783) observed that for the reduced quartic polynomial equation the sum of the roots \( x_1 + x_2 + x_3 + x_4 \) is equal to zero. For the variation of the values of the four roots, this condition provides only three degree of freedom [9]. This is so because if any three of the variables \( x \) are fixed, the fourth variable cannot be varied and has to be determined from the three variables. This suggested to Euler that the roots can be expressed in terms of three transformed variables. Using this fact, a resolvent cubic is solved that determines these three variables from which the roots are obtained.

Fathi and Sharifan [10] gave a new method for the solution of the quartic equation using three transformed variables \( u, v \) and \( s \) such that \( x = u + v + s \). Kularni [11] formulated a unified method for solving polynomial equations and demonstrated the application of the method for quadratic, cubic and quartic polynomials.

The method of deriving true roots separate from the false root using the novel Tschirnhaus transformation has been provided by Kulkarni [12]. The advantage of the Tschirnhaus transformation is that the polynomials after the transformation is easy to solve such as the form \( y^3 + A = 0 \) for cubic polynomials and \( y^4 + ay^2 + b = 0 \) for quartic polynomials. For a more simplified approach for solving quartic equations, the recent work of Okoli et al [13] can be referred to. For a unified approach of solving quadratic, cubic and quartic equations using radicals including the conditions for the occurrence of real, complex, identical and repeating roots, the paper by Ungal [14] provides sufficient explanation. Several of the solutions to quartic equations involve a resolvent cubic equation of different forms which may be seen as branches of some parent unifying method. For this, the work of Shimakov [15] casts light on establishing a unifying approach for solving quartic polynomial equations.
The methods presented in this paper provide alternative solution to cubic and quartic equations using a general equivalent polynomial that can be solved directly with minimum of algebraic manipulation. In this way, the objective of presenting this alternative method is to illustrate a simpler, more direct and intuitive procedure for solving polynomial equations.

2 Methodology

Given a cubic or quartic polynomial equation \( f(x) = 0 \), the general formula applicable to cubic and quartic equations is developed from a simple formulation of the equivalent polynomial of the form:

\[
p (g(x))^m - q (h(x))^m = f(x) = 0 \tag{1}
\]

Where \( g(x) \) and \( h(x) \) are polynomials formed in such a way that \( m \leq n \) where \( n \) is the degree of the original polynomial. The coefficients \( p \) and \( q \) are real or complex constants. For cubic equations, assuming the original equation is in reduced form \( x^3 + bx + c = 0 \), the equivalent polynomial takes the form:

\[
p(x + u)^3 - q(x + v)^3 = f(x) = 0 \tag{2}
\]

Where \( p, q, u \) and \( v \) are constants (real or complex) to be determined. In Equation (2) above, \( g(x) = x + u \); \( h(x) = x + v \) and \( m = 3 \).

For quartic polynomials, for a given equation that is in reduced form; \( f(x) = x^4 + bx^2 + cx + d = 0 \), the equivalent polynomial takes the form:

\[
[2(x^2 + ux + v)^2] - [1(x^2 + sx + r)]^2 = 3f(x) = 0 \tag{3}
\]

In Equation (3) \( p = 4 \) and \( q = 1 \) so that \( f(x) \) is multiplied by \( p-q = 4-1 = 3 \) to maintain equivalence. Other values of \( p \) and \( q \) such as \( p = 2 \) and \( q = 1 \) can be used. However, the solution, contains the square root of 2 as an added complexity and can be easily avoided by the choice of \( p = 4 \) and \( q = 1 \). The sections that follow describe the procedure for establishing the solution using the above formulae. In the equation above, \( g(x) = x^2 + ux + v \); \( h(x) = x^2 + sx + r \); and \( m = 2 \)

2.1 Cubic Equations

Consider the cubic polynomial equation that is in reduced form which is obtained by the method of functional evaluation (Tiruneh, 2020) or variable transformation of the general cubic polynomial equation as given in the introduction section. Let the reduced polynomial be given by the form:

\[
x^3 + bx + c = 0
\]

Consider an equivalent polynomial again given by Equation (2), i.e.,

\[
p(x + u)^3 - q(x + v)^3 = f(x) = x^3 + bx + c = 0
\]

The solution for \( x \) in terms of coefficients of the polynomial can be directly and easily solved without intermediate variable substitution as:

\[
x = - \frac{p^{1/3}u - q^{1/3}v}{p^{1/3} - q^{1/3}} \tag{4}
\]

The solution in Equation (4) is slightly simplified further by defining a variable \( R \) such that:

\[
R = \frac{p}{q}
\]
The final form of the solution for $x$ will then be:

$$x = -\frac{R^{1/3}u - v}{R^{1/3} - 1}$$  \hspace{1cm} (5)

It is now left to determine the coefficients of the equivalent form $u$, $v$, $p$ and $q$ (and hence $R = p/q$) in Equation (5). This is accomplished first be expanding the equivalent cubic form and equating the corresponding coefficients in the original reduced equation. The expansion of the equivalent form given by Equation (2) will be:

$$(p - q)x^3 + (3pu - 3qv)x^2 + (3pu^2 - 3qv^2)x + pu^3 - qv^3 = 0$$  \hspace{1cm} (6)

Comparison of the coefficients of the equivalent cubic equation in Equation (6) with the original cubic polynomial equation $x^3 + bx + c = 0$ gives the following set of equations:

$$p - q = 1$$  \hspace{1cm} (7)

$$3pu - 3qv = 0$$  \hspace{1cm} (8)

$$3pu^2 - 3qv^2 = b$$  \hspace{1cm} (9)

$$pu^3 - qv^3 = c$$  \hspace{1cm} (10)

From Equation (7) and Equation (8) one can get the following relationships:

$$q = p - 1$$

$$u = \left(\frac{p - 1}{p}\right)v$$

Substituting these two relationships in Equation (9) results in the following relationship:

$$p = \frac{v^2}{\frac{b}{3} + v^2} \quad \text{or} \quad v^2 = \frac{b}{3} \left(\frac{p}{1 - p}\right)$$

Substituting these new found relationships in Equation (10) gives:

$$v^3 = \left(\frac{p}{p - 1}\right) \cdot \left(\frac{p}{1 - 2p}\right) \cdot c$$

Finally; dividing the expression for $v^3$ by that of $v^2$:

$$\frac{v^3}{v^2} = v = \frac{\left(\frac{p}{p - 1}\right) \cdot \left(\frac{p}{1 - 2p}\right) \cdot c}{\frac{b}{3} \left(\frac{p}{1 - p}\right)} = \left(\frac{p}{2p - 1}\right) \frac{3c}{b}$$

or

$$\frac{b}{3} (2p - 1) v = pc$$
The coefficient $p$ is eliminated using the relationship:

$$p = \frac{v^2}{(\frac{b}{3}) + v^2}$$

This will give:

$$\frac{b}{3} \left( 2 * \left[ \frac{v^2}{(\frac{b}{3}) + v^2} \right] - 1 \right) v = \left( \frac{b}{3} + v^2 \right) c$$

Simplifying further gives:

$$v^2 - \left( \frac{3c}{b} \right) v - \frac{b}{3} = 0 \quad (11)$$

Equation (11) is readily solved for $v$ using the quadratic formula:

$$v = \frac{3c}{2b} \pm \sqrt{\left( \frac{3c}{2b} \right)^2 + \frac{b}{3}} \quad (12)$$

It is to be noted that the two alternate values of $v$ provide the same solution since the relationship:

$$uv = -\frac{b}{3}$$

holds true which means $u$ and $v$ are interchangeable giving the same solution to the cubic equation. Once the value of $v$ is obtained, the other coefficients are calculated using the established relationships as follows (including the expression for $v$):

$$v = \frac{3c}{2b} \pm \sqrt{\left( \frac{3c}{2b} \right)^2 + \frac{b}{3}}$$

$$p = \frac{v^2}{(\frac{b}{3}) + v^2}$$

$$q = \frac{-b/3}{(\frac{b}{3}) + v^2}$$

$$u = -\frac{b}{3v}$$

$$R = \frac{p}{q} = \frac{-3v^2}{b}$$

The values of $p$ and $q$ need not be calculated as $R$ is directly worked out from $v$ and $b$. Once the values of $u$, $v$ and $R$ are obtained the formula given earlier is used to obtained the solution for $x$ directly given by Equation (5), i.e.,

$$x = -\left[ \frac{R^{1/3}u - v}{R^{1/3} - 1} \right]$$
2.1.1 The case of repeating root (degenerating case)

The case where $R^{1/3} = 1$ is a degenerate case involving repeating roots in which $u=v$. The solution apparently looks like the denominator having a value of zero and hence division by zero is involved making it impossible to obtain the root. Fortunately, the $(R^{1/3} - 1)$ term in the solution formula cancels out and the solution will be given by $x = -v = -u$. The last example (Example 3) demonstrates this degenerate case. To show that the case $v=u$ involves a repeating root, consider the cubic polynomial equation containing repeating root $\alpha$ and the other root $\beta$ so that the equation can be written in the form:

$$f(x) = (x - \alpha)^2(x - \beta) = 0$$

By using the method of function evaluation, the equation can be reduced using the transformation $x = z + t$ so that:

$$f(t) = t^3 + \frac{f''(z)}{2} t^2 + f'(z) t + f(z) = 0$$

$$f'(z) = 2(z - \alpha)(x - \beta) + (x - \alpha)^2$$

$$f''(z) = 2(z - \alpha) + 2(x - \beta) + 2(x - \alpha) = 6z - (4\alpha + 2\beta)$$

Since $f''(z) = 0$ for the reduced case so that the $t^2$ coefficient is zero it follows that:

$$6z - (4\alpha + 2\beta) = 0; \quad z = \frac{4\alpha + 2\beta}{6} = \frac{2\alpha + \beta}{3}$$

Evaluating $f(z)$ and $f'(z)$ using this value gives:

$$f(z) = \left(\frac{2\alpha + \beta}{3} - \alpha\right)^2 \left(\frac{2\alpha + \beta}{3} - \beta\right) = \frac{2(\alpha - \beta)^3}{27}$$

$$f'(z) = 2\left(\frac{2\alpha + \beta}{3} - \alpha\right)\left(\frac{2\alpha + \beta}{3} - \beta\right) + \left(\frac{2\alpha + \beta}{3} - \alpha\right)^2 = \frac{1}{3} (\alpha - \beta)^2$$

The reduced cubic then takes the form:

$$f(t) = t^3 - \frac{1}{3} (\alpha - \beta)^2 t + 2\frac{(\alpha - \beta)^3}{27} = 0$$

So that in the equation $x^3 + bx + c = 0$:

$$b = -\frac{1}{3} (\alpha - \beta)^2; \quad c = 2\frac{(\alpha - \beta)^3}{27}$$

It can be shown that the expression for $v$ given by Equation (12)

$$v = \frac{3c}{2b} \pm \sqrt{\left(\frac{3c}{2b}\right)^2 + b \frac{b}{3}}$$
has its square root term zero which means that:

\[
\left(\frac{3c}{2b}\right)^2 = \frac{1}{9}(\alpha - \beta)^2 = -\frac{b}{3} = -\frac{1}{3}\left(-\frac{1}{3}(\alpha - \beta)^2\right) = \frac{1}{9}(\alpha - \beta)^2
\]

\[
v = \frac{3c}{2b}; \ u = -\frac{b}{3} = \frac{(\frac{3c}{2b})^2}{v} = \frac{3c}{2b} = v
\]

This shows that when the radical expression is zero \( u=v \) and the cubic polynomial equation has repeating roots.

### 2.2 Quartic Polynomials

For a given quartic polynomial equation that is in reduced form; \( f(x) = x^4 + bx^2 + cx + d = 0 \), the equivalent polynomial takes the form:

\[
\left(\frac{2x^2}{a} + ux + v\right)^2 - \left(\frac{1}{a}(x^2 + sx + r)\right)^2 = 3f(x) = 0 \quad (13)
\]

In Equation (13), \( p=4 \) and \( q=1 \) so that \( f(x) \) is multiplied by \( p-q=4-1=3 \) to maintain equivalence. The solution is simply obtained by decomposition as follows:

\[
A^2 - B^2 = (A - B)(A + B) = 0
\]

\[
A - B = 0 \; ; A + B = 0 \; or \; A = B = 0
\]

Taking \( A = 2(x^2 + ux + v) \) and \( B = (x^2 + sx + r): \)

\[
2(x^2 + ux + v) \pm (x^2 + sx + r) = 0
\]

\[
(2 \pm 1)x^2 + (2u \pm s)x + (2v \pm r) = 0
\]

The solution \( x \) will be in general form:

\[
x = \frac{-(2u \pm s) \pm \sqrt{(2u \pm s)^2 - 4(2 \pm 1)(2v \pm r)}}{2*(2 \pm 1)} \quad (14)
\]

The four values of \( x \) will be:

\[
x_1 = \frac{-(2u + s) + \sqrt{(2u + s)^2 - 4(3)(2v + r)}}{2*(3)} \quad (15)
\]

\[
x_2 = \frac{-(2u + s) - \sqrt{(2u + s)^2 - 4(3)(2v + r)}}{2*(3)} \quad (16)
\]

\[
x_3 = \frac{-(2u - s) + \sqrt{(2u - s)^2 - 4(1)(2v - r)}}{2*(1)} \quad (17)
\]

\[
x_4 = \frac{-(2u - s) - \sqrt{(2u - s)^2 - 4(1)(2v - r)}}{2*(1)} \quad (18)
\]
The undetermined coefficients of the equivalent polynomial \( u, v, s \) and \( r \) will be determined as follows. Expanding the equivalent polynomial and equating it with original polynomial gives:

\[
2(x^2 + ux + v)^2 - [1(x^2 + sx + r)]^2 = 3f(x) = 0
\]

\[
2(x^2 + ux + v)^2 - [1(x^2 + sx + r)]^2 = 3x^4 + (8u - 2s)x^3 + (4u^2 + 8v - s^2 - 2r)x^2 + (8uv - 2sr)x + 4v^2 - r^2
\]

\[
3f(x) = 3x^4 + 0x^3 + 3bx^2 + 3cx + 3d = 0
\]

Equating the coefficients term by term gives the following equations:

\[
8u - 2s = 0 \quad \text{(19)}
\]

\[
4u^2 + 8v - s^2 - 2r = 3 \quad \text{(20)}
\]

\[
8uv - 2sr = 3c \quad \text{(21)}
\]

\[
4v^2 - r^2 = 3d \quad \text{(22)}
\]

From Equation (19) one gets \( s = 4u \). Substituting \( s = 4u \) in equation (20) and equation (21) and solving for \( v \) and \( r \) simultaneously in terms of \( u \) gives:

\[
v = \frac{b}{2} + 2u^2 - \frac{c}{8u}
\]

\[
r = \frac{b}{2} + 2u^2 - \frac{c}{2u}
\]

Substituting the expressions of \( v \) and \( r \) in terms of \( u \) in Equation (22) after rearranging gives:

\[
u^6 + \frac{b}{2} u^4 + \left(\frac{b^2}{16} - \frac{d}{4}\right) u^2 - \frac{c^2}{64} = 0
\]

(23)

Defining a new variable \( w \) in Equation (23) such that \( w = u^2 \):

\[
w^3 + \frac{b}{2} w^2 + \left(\frac{b^2}{16} - \frac{d}{4}\right) w - \frac{c^2}{64} = 0
\]

(24)

Equation (24) is a cubic resolvent equation and can be solved using methods such as the one given in this paper or other established methods. It is to be noted that the resolvent cubic equations has three roots and all of the roots give the same solution for the quartic equation. Since the cubic equation should at least have one real root, it can be used where complex roots are also involved in the resolvent cubic. Otherwise as will be shown in the example below, if all the roots of the resolvent cubic are real, any of them can be used and give the same solution for the original quartic equation.

The equations that will be solved in steps are then as follows:

\[
w^3 + \frac{b}{2} w^2 + \left(\frac{b^2}{16} - \frac{d}{4}\right) w - \frac{c^2}{64} = 0
\]
Once the coefficients of the equivalent quartic \( u, v, s \) and \( r \) are determined through the above procedure, the solution of the original reduced quartic equation is worked out as given by the Equations (15) – Equation (18) above.

**3 Results and Discussion**

Examples of application of the new alternative formula for the cubic and quartic equations are demonstrated in this section with examples.

**3.1 Cubic Equation**

**Example 3.1A (real roots):** Given \( f(x) = x^3 - 6x + 4 = 0 \)

The above equation is an example in which all the three roots of the cubic equations are real numbers. For the given equation, \( b = -6 \) and \( c = 4 \). The solution for \( v \) is obtained as:

\[
v = \frac{3c}{2b} \pm \sqrt{\left(\frac{3c}{2b}\right)^2 + \frac{b}{3}} = \frac{3 \times 4}{2 \times (-6)} \pm \sqrt{\left(\frac{3 \times 4}{2 \times (-6)}\right)^2 + \frac{-6}{3}} = -1 \pm i
\]

It is interesting to note that the expression has the same sign (<0) as the well-known discriminant for cubic equations, i.e.,

\[
D = \left(\frac{c}{2}\right)^2 + \left(\frac{b}{3}\right)^3 = \left(\frac{4}{2}\right)^2 + \left(\frac{-6}{3}\right)^3 = 4 - 8 = -4 < 0
\]

Therefore, all the solutions are real since \( D < 0 \).

Because of interchangeability of \( u \) and \( v \) take the following value \( v \) for convenience:

\[
v = -1 + i
\]

\[
u = \frac{-b}{3v} = \frac{-(-6)}{3 \times (-1 + i)} = \frac{2}{i - 1} = \frac{2}{i - 1} \times \frac{(i + 1)}{(i + 1)} = -i - 1
\]

The value of \( R \) can be directly computed as:

\[
R = \frac{-3u^2}{b} = \frac{-3 \times (-1 + i)^2}{-6} = \frac{1}{2} (-2i) = -i
\]

\( R \) can be expressed in trigonometric form as:

\[
R = 1 \times \left[\cos \left(-\frac{\pi}{2}\right) + i \sin \left(-\frac{\pi}{2}\right)\right]
\]
The three cubic roots of $R$ will be:

$$R_1^{1/3} = 1^{1/3} [\cos \left( -\frac{\pi}{2} \right) + i \sin \left( -\frac{\pi}{2} \right)] = \cos \left( -\frac{\pi}{6} \right) + i \sin \left( -\frac{\pi}{6} \right) = \frac{\sqrt{3}}{2} - \frac{1}{2}i$$

$$R_2^{1/3} = 1^{1/3} \left[ \cos \left( -\frac{\pi}{6} + \frac{2\pi}{3} \right) + i \sin \left( -\frac{\pi}{6} + \frac{2\pi}{3} \right) \right] = \cos \left( \frac{\pi}{2} \right) + i \sin \left( \frac{\pi}{2} \right) = i$$

$$R_3^{1/3} = 1^{1/3} \left[ \cos \left( -\frac{\pi}{6} + \frac{4\pi}{3} \right) + i \sin \left( -\frac{\pi}{6} + \frac{4\pi}{3} \right) \right] = \cos \left( \frac{7\pi}{6} \right) + i \sin \left( \frac{7\pi}{6} \right)$$

$$= -\frac{\sqrt{3}}{2} - \frac{1}{2}i$$

Using the second root $R_2^{1/3} = i$ gives:

$$x = - \left[ \frac{R_2^{1/3} - v}{R_2^{1/3} - 1} \right] = - \left[ \frac{(i)(-1-i) - (-1+i)}{i-1} \right] = - \left[ \frac{2 * (1-i)}{i-1} \right] = 2$$

Similarly using the other cubic roots of $R$ on gets:

For $R_1 = \frac{\sqrt{3}}{2} - \frac{1}{2}i$

$$x = - \left[ \frac{R_1^{1/3} - v}{R_1^{1/3} - 1} \right] = -1 - \sqrt{3}$$

For $R_3 = \frac{\sqrt{3}}{2} - \frac{1}{2}i$

$$x = - \left[ \frac{R_3^{1/3} - v}{R_3^{1/3} - 1} \right] = -1 + \sqrt{3}$$

Therefore the three roots of the cubic equation $x^3 - 6x + 4 = 0$ are:

$$x = \{2, -1 - \sqrt{3}, -1 + \sqrt{3}\}$$

Example 3.1B (With complex roots) 2: Given $f(x) = x^3 - 2x + 4 = 0$

The above equation has two complex roots and one real root. For the given equation, $b = -2$ and $c = 4$.

$$v = \frac{3c}{2b} = \frac{3 * 4}{2 * (-2)} = \frac{3}{2} \frac{4}{2} \frac{(-2)}{(-2)} = \frac{3 * 4}{2 * (-2)^2} = -3 \pm \frac{5 \sqrt{3}}{3}$$

The discriminant for cubic equations, i.e.,

$$D = \left( \frac{c}{2} \right)^2 + \left( \frac{b}{3} \right)^3 = \left( \frac{4}{2} \right)^2 + \left( \frac{-2}{3} \right)^3 = 4 - \frac{8}{27} > 0$$
Since $D>0$, the solution contains complex numbers.

Because of interchangeability of $u$ and $v$, take the following value $v$ for convenience:

$$v = -3 + \frac{5\sqrt{3}}{3} = -0.113248654$$

$$u = -\frac{b}{3v} = -3 - \frac{5\sqrt{3}}{3} = -5.8867513$$

The value of $R$ can be directly computed as:

$$R = \frac{-3v^2}{b} = \frac{-3\left(-0.113248654\right)^2}{-2} = 0.019237886$$

The real cube root of $R$ will be:

$$R^{1/3} = (0.019237886)^{1/3} = 0.267949192$$

The first real root will be:

$$x = -\left(\frac{1}{R^{3/2} - 1}\right) = -\left[\frac{(0.267949192)(-5.8867513) - (-0.113248654)}{0.267949192 - 1}\right] = -2$$

The second and third complex root can be easily computed using division:

$$\frac{x^3 - 2x + 4}{x + 2} = x^2 - 2x + 2$$

The root of the above quadratic equation can be worked out as:

$$x = z \pm \sqrt{-f(z)} ; z = -\frac{2}{2} = 1$$

$$f(z) = f(1) = (1)^2 - 2(1) + 2 = 1$$

$$x = 1 \pm \sqrt{-1} = 1 \pm i$$

The three roots of the cubic equation are, therefore,

$$x = \{-2 , 1 + i , 1 - i \}$$

**Example 3.1C (Degenerate case – Double roots):**

Given $f(x) = x^3 - 3x - 2 = 0$

As will be shown below, this is a degenerate case where $u = v$, containing repeating root and in which the solution can be easily worked out by elimination of the common factor containing the $R^{1/3}$ term in the solution. Otherwise, the value of $R$ is actually equal to one and the denominator of the solution becomes zero.

For the given equation, $b = -3$ and $c = -2$:

$$v = \frac{3c}{2b} \pm \left[\frac{\sqrt{\left(\frac{3c}{2b}\right)^2 + \frac{b}{3}}}{\frac{3}{2}} \right] = \frac{3 \times (-2)}{2 \times (-3)} \pm \frac{-3}{3} = 1 \pm 0 = 1$$
The discriminant for cubic equations, i.e.,

\[ D = \left( \frac{b}{3} \right)^2 + \left( \frac{c}{2} \right)^3 = \left( \frac{-21}{2} \right)^2 + \left( \frac{-3}{3} \right)^3 = 1 - 1 = 0 \]

Since \( D = 0 \), the solution contains repeating root. It can be seen that \( u = v \) since the term in the square root is zero. This is verified further from the formula for \( u \) as follows:

\[ u = \frac{-b}{3v} = \frac{-(-3)}{3 \times 1} = 1 = v \]

For the \( u=v \) case, the solution eliminates the \( R \) term, fortunately, as shown below. Otherwise the denominator would have become zero and hence it would have been impossible to use the formula for the solution for \( x \) directly.

\[ x = -\left[ \frac{\frac{1}{R^3} - v}{\frac{1}{R^3} - 1} \right] = -\left[ \frac{\frac{1}{R^3} - v}{\frac{1}{R^3} - 1} \right] = -v \]

Therefore, one root of the equation will be \( x = -v = -1 \).

The second and third complex roots can be easily computed using division:

\[ \frac{x^3 - 3x - 2}{x + 1} = x^2 - x - 2 \]

The root of the above quadratic equation can be worked out as:

\[ x = z \pm \sqrt{-f(z)} \quad ; \quad z = -\frac{1}{2} = \frac{1}{2} \]

\[ f(z) = f\left( \frac{1}{2} \right) = \left( \frac{1}{2} \right)^2 - \left( \frac{1}{2} \right) - 2 = -\frac{9}{4} \]

\[ x = \frac{1}{2} \pm \sqrt{-\left( \frac{-9}{4} \right)} = \frac{1}{2} \pm \frac{3}{2} = \{2, -1\} \]

The three roots of the cubic equation are, therefore,

\[ x = \{-1, -1, 2\} \]

Apparently the degenerate case has a repeating root \( x = -1 \).

### 3.2 Quartic Equations

**Example:** Given a quartic equation:

\[ x^4 - x^3 - 19x^2 - 11x + 30 = 0 \]

The solution to the above quartic equation is:

\[ x = \{-3, -2, 1, 5\} \]
The given quartic equation can be reduced using the method of function evaluation (Tiruneh, 2020). This method of function evaluation can also be derived from binomial expansion as follows. Define variables $z$ and $t$ such that:

$$x = z + t$$

The value of $z$ is chosen such that the resulting quartic polynomial equation is in reduced form in terms of the variable $t$. Using the binomial expansion, the original equation can be expressed in terms of $z$ and $t$ as follows:

$$f(t) = t^4 + \frac{f''''(z)}{3!} t^3 + \frac{f''''(z)}{2!} t^2 + f'(z)t + f(z) = 0$$

Since the coefficient of $t^3$ has to be zero in the reduced form, it follows that:

$$f''''(z) = 0$$

$$24z - 6 = 0$$

$$z = \frac{1}{4}$$

Evaluating the other coefficients:

$$f(z) = f \left( \frac{1}{4} \right) = \left( \frac{1}{4} \right)^4 - \left( \frac{1}{4} \right)^3 - 19 \left( \frac{1}{4} \right)^2 - 11 \left( \frac{1}{4} \right) + 30 = 26.05078125$$

$$f'(z) = f' \left( \frac{1}{4} \right) = 4z^3 - 3z^2 - 38z - 11 = -20.625$$

$$f''(z) = f'' \left( \frac{1}{4} \right) = 12z^2 - 6z - 38 = -38.75$$

The reduced quartic then becomes:

$$f(t) = t^4 + \frac{f''''(z)}{3!} t^3 + \frac{f''''(z)}{2!} t^2 + f'(z)t + f(z) = 0$$

$$f(t) = t^4 - 19.375 t^2 - 20.625 t + 26.05078125 = 0$$

For the coefficients of the resolvent cubic equation:

$$\frac{b}{2} = \frac{-19.375}{2} = -9.6875;$$

$$\frac{b^2}{16} - \frac{d}{4} = \frac{(-19.375)^2}{16} - \frac{26.05078125}{4} = 16.94921875$$

$$\frac{-c^2}{64} = \frac{(-20.625)^2}{64} = -6.646728516$$

The resolvent cubic will then be:

$$w^3 + \frac{b}{2} w^2 + \left( \frac{b^2}{16} - \frac{d}{4} \right) w - \frac{c^2}{64} = 0$$
The solution of the above resolvent cubic using the method developed in this paper is found to be:

\[ w = \{0.5625, \quad 1.5625, \quad 7.5625\} \]

Any of these three roots can be used to proceed with finding the solution of the quartic equation as all of them give the same answer. For convenience the first solution is chosen, i.e., \( w = 0.5625 \).

\[ u = \pm \sqrt{w} = \pm \sqrt{0.5625} = \pm 0.75 \]

Take the positive root \( u = 0.75 \) (again the negative root will also provide same solution).

\[ u = 0.75; \quad s = 4u = 4 \times 0.75 = 3 \]

\[ v = \frac{b}{2} + 2u^2 - \frac{c}{8u} = \frac{-19.375}{2} + 2(0.75)^2 - \frac{(-20.625)}{8(0.75)} = -5.125 \]

\[ r = \frac{b}{2} + 2u^2 - \frac{c}{2u} = \frac{-19.375}{2} + 2(0.75)^2 - \frac{(-20.625)}{2(0.75)} = +5.1875 \]

Now the solution in terms of \( t \):

\[ 2u + s = 2(0.75) + 3 = 4.5 \quad \text{and} \quad 2v + r = 2(-5.125) + 5.1875 = -5.0625 \]

\[ t_1 = \frac{-(2u + s) + \sqrt{(2u + s)^2 - 4(3)(2v + r)}}{2 \times (3)} \]

\[ t_1 = \frac{-(4.5) + \sqrt{(4.5)^2 - 12(-5.0625)}}{6} = 0.75 \]

\[ t_2 = \frac{-(2u + s) - \sqrt{(2u + s)^2 - 4(3)(2v + r)}}{2 \times (3)} \]

\[ t_2 = \frac{-(4.5) - \sqrt{(4.5)^2 - 12(-5.0625)}}{6} = -2.25 \]

\[ 2u - s = 2(0.75) - 3 = -1.5 \quad \text{and} \quad 2v - r = 2(-5.125) - 5.1875 = -15.4375 \]

\[ t_3 = \frac{-(2u - s) + \sqrt{(2u - s)^2 - 4(3)(2v - r)}}{2 \times (1)} \]

\[ t_3 = \frac{-(1.5) + \sqrt{(-1.5)^2 - 4(-15.4375)}}{2} = 4.75 \]

\[ t_4 = \frac{-(2u - s) - \sqrt{(2u - s)^2 - 4(3)(2v - r)}}{2 \times (1)} \]

\[ t_4 = \frac{-(1.5) - \sqrt{(-1.5)^2 - 4(-15.4375)}}{2} = -3.25 \]
Finally converting the solutions to the original value of $x$ using $x = z + t$ and $z = \frac{1}{4}$

\[
x_1 = z + t_1 = \frac{1}{4} + 0.75 = 1
\]

\[
x_2 = z + t_2 = \frac{1}{4} - 2.25 = -2
\]

\[
x_3 = z + t_3 = \frac{1}{4} + 4.75 = 5
\]

\[
x_4 = z + t_4 = \frac{1}{4} \pm 3.25 = -3
\]

Giving the solution in terms of $x$ of:

\[x = \{-3, -2, 1.5\}\]

The examples given for the cubic and quartic equations demonstrated a more intuitive and easy procedure for decomposing the original equations as a difference of two symmetric polynomials. From which the solution can be worked out.

4 Conclusion

Two alternative procedures for the analytical solution of cubic and quartic equations have been presented and demonstrated with worked examples in this paper. For the cubic equation the intuitive nature of the solution is evident as the solution is derived through a direct procedure without involving intermediate variable substitution. The historical formulas by both Cardan and Viete indeed provide an amazing discovery of the hidden symmetry within the cubic equation that allows reduction of the equation into a quadratic form through a substituted variables in $t$ where $x = t - \frac{b}{t}$ and further $s$ whereby $s = t^2$. The formula proposed in this paper for cubic equations on the other hand involves explicit solution of an equivalent cubic that is formed directly in terms of the original variable $x$. As such, the formula is intuitive and simple to derive or understand as well as apply. For the quartic equations the same decomposition form is used as that of the cubic using two quadratic polynomials that have symmetric form thus making it easy to develop the solution as well as solve the equations. Overall, the proposed alternative formulae for both cubic and quartic equations provide an interesting reflection of the many ways in which polynomial equations can be solved analytically.

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References


Accessed: 5 March, 2021


Accessed: 1 March, 2021

Accessed: 25 February, 2021


