Oscillation Criteria for a Class of Second-Order Differential Equation with Neutral Term

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Authors’ contributions
This work was carried out in collaboration between both authors. YPZ designed the study, performed the statistical analysis, wrote the protocol, wrote the first draft of the manuscript and managed the analyses of the study. FH managed the literature searches. All authors read and approved the final manuscript.

Article Information
DOI: 10.9734/JAMCS/2021/v36i230340

Editor(s):
(1) Dr. Raducanu Razvan, Alexandru Ioan Cuza University, Romania.

Reviewers:
(1) Sheng Qi, State Key Laboratory of NBC Protection for Civilian, Logistical Engineering University, China.
(2) Alexandre de Macêdo Wahrhaftig, Federal University of Bahia, Brazil.

Complete Peer review History: http://www.sdiarticle4.com/review-history/66383

Received: 01 January 2021
Accepted: 03 March 2021
Published: 31 March 2021

Original Research Article

Abstract
This paper is concerned with oscillation criteria for a class of second-order differential equation with neutral term. We obtain sufficient conditions for the oscillation of this equation by using Riccati transformation and some analytical skill.

Keywords: Differential equation; oscillation; second-order; riccati transformation.

2010 Mathematics Subject Classification: 34K10, 34K11.

1 Introduction
In this paper, we consider oscillation criteria for a class of second-order differential equation with neutral term.
\begin{equation}
(r(t)z'(t))' + q(t)f(x(g(t))) = 0, \quad t \geq t_0 > 0
\end{equation}

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where $z(t) = x(t) + p(t)x(\delta(t))$. In the following, it is always assume

(C1) $p(t) \in C([t_0, \infty), R^+)$, $0 \leq p(t) < 1$, $f \in C(R, R)$, $\frac{f(v)}{v} \geq \lambda$, for all $v \neq 0$, and for some $\lambda > 0$ is constant. $\alpha$ is a quotient of odd positive integers.

(C2) $r(t) \in C^1([t_0, \infty), R^+)$, $r'(t) \geq 0$, $g(t), g(t) \in C([t_0, \infty), R^+)$, and $g(t) \leq t$, $g'(t) \geq 0$, $\lim_{t \to \infty} g(t) = \infty$.

(C3) $\delta(t) \in C([t_0, \infty), R)$, $\delta(t) \leq t$, $\lim_{t \to \infty} \delta(t) = \infty$.

By a solution of equation (1.1), we mean a continuous function $x(t)$ defined on an interval $[t_0, \infty)$ such that $(r(t)z'(t))'$ is continuously differentiable satisfies equation (1.1), we assume that equation (1.1) is oscillatory if it is arbitrarily large zeros on $[t_0, \infty)$, otherwise, it is called nonoscillatory. We say equation (1.1) is oscillatory if all its continuinble solutions are oscillatory.

In what follows, we consider only proper solution of the equation (1.1) which are defined for all large $t$. More and more people are interested in oscillatory criteria to be shown [1-11]. Agarwal just studied oscillation properties of the linear equation in [2], the oscillation properties of second order superlinear differential equation were considered in [1], but our principal goal in this paper is to derive new oscillation criteria for nonlinear equation (1.1) under the conditions,

$$\int_{t_0}^{\infty} \frac{1}{r(s)} \, ds = \infty, \quad t \geq t_0$$

$$\int_{t_0}^{\infty} \frac{1}{r(s)} \, ds < \infty, \quad t \geq t_0$$

For simplicity, we introduce the following notations:

$$R(t) = \int_{t_0}^{t} \frac{1}{r(s)} \, ds, \quad U(t) = \int_{t_0}^{\infty} \frac{1}{r(s)} \, ds, \quad Q(t) = 1 - \frac{p(t)U(\delta(t))}{U(t)}.$$

Without loss of generality, we only give proofs for the case of eventually positive solutions since the proofs for the eventually negative solutions would be similar.

**Lemma 1.** Assume that (C1)-(C3) hold. Let $x(t)$ is eventually positive solution of equation (1.1), then there exists a $t_1$ for $t > t_1 \geq t_0$, such that $z(t)$ has only the following two cases:

**case 1** $z(t) > 0$, $z'(t) > 0$, $(r(t)z'(t))' \leq 0$;

**case 2** $z(t) > 0$, $z'(t) < 0$, $(r(t)z'(t))' \leq 0$.

**Lemma 2.** Assume that $x(t)$ is a eventually positive solutions of equation (1.1) satisfying case 2 of Lemma 1, then

$$x(t) \geq Q(t)z(t).$$

$$z(t) \geq C U(t).$$

$C > 0$ is a constant and all $t \geq t_1 \geq t_0$.

The proof method is similar to Lemma 3 in reference 1, so it is omitted.

## 2 Oscillation theorems

**Theorem 1.** Assume (C1)-(C3) and (1.2) hold. If

$$\int_{t_1}^{\infty} q(s)(1 - p(g(s)))^\alpha \, ds = \infty.$$  

then every solution of equation (1.1) is oscillatory.
Taking \( \lim as \) \( z(t) \). Case 2. In this case, we find 
\[
x(t) = z(t) - p(t)x(\delta(t)) \geq z(t) - p(t)z(\delta(t)) \geq z(t)(1 - p(t)),
\]
(2.2) 

From equation (1.1) and (C1), we have 
\[
(r(t)z'(t))^\prime + \lambda(q(t))z^\alpha(g(t)) \leq 0,
\]
using (2.2), we obtain 
\[
(r(t)z'(t))^\prime + \lambda(q(t))z^\alpha(g(t))(1 - p(g(t)))^\alpha \leq 0,
\]
Integrating the inequality (2.4) from \( t_1 \) to \( t \), we get 
\[
z^\alpha(g(t_1)) \int_{t_1}^{t} \lambda(q(s))(1 - p(g(s)))^\alpha ds \leq \int_{t_1}^{t} \lambda(s)z^\alpha(g(s))(1 - p(g(s)))^\alpha ds \leq r(t_1)z'(t_1) - r(t)z'(t) \leq r(t_1)z'(t_1).
\]
The last inequality contradicts (2.1) as \( t \to \infty \).

**Theorem 2.** Let \( \alpha \geq 1 \), Assume (C1)-(C3),(1.3),(2.1) and \( Q(t) > 0 \) hold. and 
\[
\lim_{t \to \infty} \int_{t_1}^{t} \lambda(q(s))Q^\alpha(g(s))U^\alpha(s) - \frac{\alpha}{L^2} \int_{t_1}^{t} r(s)U(s) ds = \infty.
\]
(2.5) 

for any positive constant \( L_2 \), then every solution of equation (1.1) is oscillatory.

**Proof.** Assume \( z(t) \) is a eventually positive solutions of equation (1.1), then there exists \( t_1 \geq t_0 \) such that \( x(t) > 0, x(\delta(t)) > 0, x(g(t)) > 0 \) for all \( t \geq t_1 \), from Lemma 1, since case 2 can’t hold when condition (1.2) holds, we only consider case 1. We find \( z(t) \geq x(t) \)

\[
x(t) = z(t) - p(t)x(\delta(t)) \geq z(t) - p(t)z(\delta(t)) \geq z(t)(1 - p(t)),
\]
(2.2) 

Using (2.2), we obtain 
\[
(r(t)z'(t))^\prime + \lambda(q(t))z^\alpha(g(t))(1 - p(g(t)))^\alpha \leq 0,
\]
Integrating the inequality (2.4) from \( t_1 \) to \( t \), we get 
\[
z^\alpha(g(t_1)) \int_{t_1}^{t} \lambda(q(s))(1 - p(g(s)))^\alpha ds \leq \int_{t_1}^{t} \lambda(s)z^\alpha(g(s))(1 - p(g(s)))^\alpha ds \leq r(t_1)z'(t_1) - r(t)z'(t) \leq r(t_1)z'(t_1).
\]

Then \( w(t) > 0 \) for all \( t \geq t_1 \), using (2.4) 
\[
w'(t) = \frac{(r(t)z'(t))^\prime}{z^\alpha(g(t))} - \frac{\alpha g'(t)r(t)z'(t)z'(g(t))}{z^\alpha(g(t))} \leq -\lambda(q(t))(1 - p(g(t)))^\alpha - \frac{\alpha z^{\alpha-1}(g(t))g'(t)w^2(t)}{r(t)},
\]
using \( 0 < L < z(g(t_1)) \) for \( t \geq t_1 \), we get 
\[
w'(t) \leq -\lambda(q(t))(1 - p(g(t)))^\alpha - \frac{\alpha L^{\alpha-1}g'(t)w^2(t)}{r(t)} \leq -\lambda(q(t))(1 - p(g(t)))^\alpha.
\]
Integrating the inequality (2.6) from \( t_1 \) to \( t \), we get 
\[
\int_{t_1}^{t} \lambda(q(s))(1 - p(g(s)))^\alpha ds \leq w(t_1) - w(t).
\]
Taking \( \lim as \) \( t \to \infty \) in the last inequality, we obtain a contradiction to (2.1).

Case 2. In this case, we find \( z'(t) < 0 \). set 
\[
u(t) = \frac{r(t)z'(t)}{z^\alpha(t)}.
\]
(2.8) 

Then \( u(t) < 0 \) for all \( t \geq t_1 \), from the equation (1.1), we have 
\[
(r(t)z'(t))^\prime = -q(t)f(x(g(t))) \leq 0.
\]
so \( z'(s) \leq \frac{r(t)z'(t)}{r(s)} \) for all \( s \geq t \). Integrating the last inequality from \( t \) to \( k \), we get 
\[
z(k) - z(t) \leq r(t)z'(t) \int_{t}^{k} \frac{1}{r(s)} ds.
\]
Proof. Case 1. In this case, we find such that

\[
-x(t) > 0, t \geq t_1
\]

Therefore

\[
\frac{-r(t)\dot{z}(t)(-r(t)\dot{z}(t))^{\alpha-1}U^\alpha(t)}{z^\alpha(t)} \leq 1, t \geq t_1
\]

From \(-r(t)\dot{z}(t) > 0, (2.8)\) and (2.9), we find

\[
\frac{-1}{L_2^{\alpha-1}} \leq u(t)U^\alpha(t) \leq 0,
\]

where \(L_2\) is a constant and such that \(0 < L_2 \leq -r(t_1)\dot{z}(t_1)\) for \(t \geq t_1\). On the other hand, from (1.4) and (1.5)

\[
u' = \frac{(r(t)\dot{z}(t))'}{z^\alpha(t)} - \frac{a(t)\dot{z}(t)\dot{z}(t)}{z^{\alpha+1}(t)} \leq -\lambda q(t)Q^\alpha(g(t)) - \frac{a(CU(t))^{\alpha-1}u^2(t)}{r(t)},
\]

\[
u' \leq -\lambda q(t)Q^\alpha(g(t)).
\]

Multiplying both sides of (2.11) by \(U^\alpha(t)\) and integrating it from \(t_1\) to \(t\), we have

\[
\int_{t_1}^{t} u'(s)U^\alpha(s)ds + \int_{t_1}^{t} \lambda q(s)Q^\alpha(g(s))U^\alpha(s)ds \leq 0,
\]

\[
u(t)U^\alpha(t) - u(t_1)U^\alpha(t_1) - \int_{t_1}^{t} u(s)duU^\alpha(s)ds + \int_{t_1}^{t} \lambda q(s)Q^\alpha(g(s))U^\alpha(s)ds \leq 0,
\]

\[
u(t)U^\alpha(t) - u(t_1)U^\alpha(t_1) - \int_{t_1}^{t} a(s)U^{\alpha-1}(s)U'(s)ds + \int_{t_1}^{t} \lambda q(s)Q^\alpha(g(s))U^\alpha(s)ds \leq 0,
\]

\[
u(t)U^\alpha(t) - u(t_1)U^\alpha(t_1) + \int_{t_1}^{t} a(s)U^{\alpha-1}(s)\frac{1}{r(s)}ds + \int_{t_1}^{t} \lambda q(s)Q^\alpha(g(s))U^\alpha(s)ds \leq 0,
\]

using (2.10),

\[
u(t)U^\alpha(t) - u(t_1)U^\alpha(t_1) - \frac{1}{L_2^{\alpha-1}} \int_{t_1}^{t} \frac{\alpha}{r(s)}U^\alpha(s)ds + \int_{t_1}^{t} \lambda q(s)Q^\alpha(g(s))U^\alpha(s)ds \leq 0,
\]

\[
\int_{t_1}^{t} (\lambda q(s)Q^\alpha(g(s))U^\alpha(s) - \frac{\alpha}{L_2^{\alpha-1}r(s)}U^\alpha(s))ds \leq u(t_1)U^\alpha(t_1) + L_2^{-\alpha}.
\]

Taking \( \limsup \) as \( t \to \infty \) in the last inequality, we obtain a contradiction to (2.5).

**Theorem 3.** Let \( \alpha < 1 \), Assume \((C1)-(C3),(1.3),(2.1)\) and \( Q(t) > 0 \) hold, and

\[
limit_{t \to \infty} \int_{t_1}^{t} (\lambda q(s)Q^\alpha(g(s))U^\alpha(s) - \frac{1}{U(s)r(s)})ds = \infty.
\]

for any positive constant \( L_3 \), then every solution of equation (1.1) is oscillatory.

**Proof.** Assume \( x(t) \) is a eventually positive solutions of equation (1.1), then there exists \( t_1 \geq t_0 \) such that \( x(t) > 0, x(\delta(t)) > 0, x(g(t)) > 0 \) for all \( t \geq t_1 \), we get \( z(t) > 0 \).

Case 1. In this case, we find \( z''(t) > 0 \). Set

\[
w(t) = \frac{r(t)\dot{z}(t)}{z^\alpha(g(t))}.
\]
Then \( w(t) > 0 \) for all \( t \geq t_1 \), using (2.4),
\[
    w'(t) = \frac{(r(t)z'(t))'}{z^2(g(t))} - \frac{\alpha g'(t)r(t)z'(t)z'(g(t))}{z^{\alpha+1}(g(t))} \leq -\lambda q(t)(1 - p(g(t)))^\alpha.
\] (2.13)

Integrating the inequality (2.13) from \( t_1 \) to \( t \), we get
\[
\int_{t_1}^{t} \lambda q(s)(1 - p(g(s)))^\alpha ds \leq w(t_1) - w(t).
\]

Taking \( \lim \) as \( t \to \infty \) in the last inequality, we obtain a contradiction to (2.1).

Case 2. In this case, we find \( z'(t) < 0 \). set
\[
    u(t) = \frac{r(t)z'(t)}{z(t)}.
\]

Then \( u(t) < 0 \) for all \( t \geq t_1 \), from (1.4)
\[
    u'(t) \leq -\lambda q(t)Q^\alpha(g(t))z^{\alpha-1}(g(t)) - \frac{u^2(t)}{r(t)} \leq -\lambda q(t)Q^\alpha(g(t))L_3^{\alpha-1} - \frac{u^2(t)}{r(t)},
\] (2.14)
\[
    u'(t) \leq -\lambda q(t)Q^\alpha(g(t))L_3^{\alpha-1},
\] (2.15)

where \( L_3 \) is a constant and such that \( 0 < L_3 \leq z(g(t_1)) \) for \( t \geq t_1 \)

Multiplying both sides of (2.15) by \( U(t) \) and integrating it from \( t_1 \) to \( t \), we have
\[
\int_{t_1}^{t} u(s)U(s)ds + \int_{t_1}^{t} \lambda q(s)Q^\alpha(g(s))L_3^{\alpha-1}U(s)ds \leq 0,
\]
\[
u(t)U(t) - u(t_1)U(t_1) - \int_{t_1}^{t} u(s)dU(s) + \int_{t_1}^{t} \lambda q(s)Q^\alpha(g(s))L_3^{\alpha-1}U(s)ds \leq 0,
\]
\[
u(t)U(t) - u(t_1)U(t_1) + \int_{t_1}^{t} \frac{u(s)}{r(s)}ds + \int_{t_1}^{t} \lambda q(s)Q^\alpha(g(s))L_3^{\alpha-1}U(s)ds \leq 0,
\]
using (2.9),
\[
u(t)U(t) - u(t_1)U(t_1) - \int_{t_1}^{t} \frac{1}{U(s)r(s)}ds + \int_{t_1}^{t} \lambda q(s)Q^\alpha(g(s))L_3^{\alpha-1}U(s)ds \leq 0,
\]
\[
\int_{t_1}^{t} (\lambda L_3^{\alpha-1}q(s)Q^\alpha(g(s))U(s) - \frac{1}{U(s)r(s)})ds \leq 1 + u(t_1)U(t_1).
\]

Taking \( \lim \) as \( t \to \infty \) in the last inequality, we obtain a contradiction to (2.12).

3 Conclusion

In this paper, we have introduced oscillation criteria of the equation (1.1). We obtain sufficient conditions for the oscillation of the equation by using Riccati transformation and some analytical skill. We can find oscillation criteria of higher-order neutral differential equation in the future work.

Acknowledgments

This work was supported Qinghai Nationalities University (Nos. 2021XJGH09).
Competing Interests
Authors have declared that no competing interests exist.

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