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Nonexistence of Global Solutions to A Semilinear Wave Equation with Scale Invariant Damping

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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Abstract

We obtain a blowup result for solutions to a semilinear wave equation with scale-invariant dissipation. We perform a change of variables that transforms our starting equation into a Generalized Tricomi equation, then apply Kato's lemma, we can prove a blowup result for solutions to the transformed equation under some assumptions on the initial data. In the critical case, we use the fundamental solutions of the Generalized Tricomi equation to modify Kato's lemma to deal with it.

Keywords: Semilinear wave equations; tricomi equation; structural damping; finite time blow up; critical power.

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1 Introduction

In this paper, we study the blowup of solutions to Cauchy problem for a semilinear wave equation with scale-invariant damping

$$\begin{cases} v_{tt} - \Delta v + \dfrac{\mu}{1+t} v_t = (1+t)^{-\alpha}|v|^p, & \text{in} \quad (0,\infty)\times\mathbb{R}^n, \\ v(0,x) = v_0(x), & x\in\mathbb{R}^n, \\ v_t(0,x) = v_1(x), & x\in\mathbb{R}^n, \end{cases} \tag{1.1}$$

where $\mu \geq 0,\ \alpha \geq 0,\ n \geq 1$ and $p > 1$.

Damped wave equations are known as models describing the voltage and the current on an electrical transmission line with a resistance. It is also derived as a modified heat conduction equation from the heat balance law and the so-called Cattaneo-Vernotte law instead of the usual Fourier law (cf. [1]). The term $b(t)v_t$ is called the damping term, which prevents the motion of the wave and reduces its energy, and the coefficient $b(t)$ represents the strength of the damping. From a mathematical point of view, it is an interesting problem to study how the damping term affects the properties of the solution. In this case we are dealing with a scale-invariant damping which is a separating threshold between effective and non-effective dissipations(cf.[2, 3]). We are interested in studying the effect of the damping term on the blowup to Cauchy problem (1.1).

Before we state the content of this paper in detail, we recall a number of related results. If $\mu = \alpha = 0$, Eq.(1.1) becomes the classical wave equation

$$\begin{cases} v_{tt} - \Delta v = |v|^p, & \text{in} \quad (0,\infty)\times\mathbb{R}^n, \\ v(x,0) = v_0(x), & x\in\mathbb{R}^n, \\ v_t(x,0) = v_1(x), & x\in\mathbb{R}^n, \end{cases} \tag{1.2}$$

and the $p_S(n)$ is the positive root of the quadratic equation

$$(n-1)p^2 - (n+1)p - 2 = 0, \quad n \geq 2, \tag{1.3}$$

if $n = 1$, we set $p_S(1) = \infty$. There are lots of literatures about Cauchy problem (1.2). We list some but may be not all of them, i.e., [4, 5, 6, 7]. Based on these known results, we may know that $p_S(n)$ is the critical exponent of Cauchy problem (1.2), if $1 < p \leq p_S(n)$, then the solutions with nonnegative initial data will blow up in finite time; if $p > p_S(n)$, then the solutions with small initial data values exist for all time.

If $\alpha = 0$, Eq.(1.1) becomes a semilinear wave equation with scale-invariant dissipation

$$\begin{cases} v_{tt} - \Delta v + \dfrac{\mu}{1+t} v_t = |v|^p, & \text{in} \quad (0,\infty)\times\mathbb{R}^n, \\ v(0,x) = v_0(x), & x\in\mathbb{R}^n, \\ v_t(0,x) = v_1(x), & x\in\mathbb{R}^n. \end{cases} \tag{1.4}$$

D'Abbicco [8] have showed that the critical power is $p_F(n)$ when

$$\mu \geq \begin{cases} \dfrac{5}{3}, & \text{if} \quad n = 1, \\ 3, & \text{if} \quad n = 2, \\ n+2, & \text{if} \quad n \geq 3. \end{cases} \tag{1.5}$$

Wakasugi [9] had obtained a blowup result, if

$$\begin{cases} 1 < p \leq p_F(n), & \mu > 1, \\ 1 < p \leq p_F(n+\mu-1), & 0 < \mu \leq 1, \end{cases} \tag{1.6}$$

where $p_F(n) = 1 + \frac{2}{n}$ see [10]. When $\mu = 2$, D'Abbicco-Lucente-Reissig [11] had got

$$p_c(n) = \max\{p_F(n), p_S(n+2)\}, \quad n \geq 2. \tag{1.7}$$

After [11], where the global existence of small data solutions is proved when $p > p_c(n)$ for $n = 2$, 3, in [12, 13] the odd dimensional case and the even dimensional case, respectively, are studied in the radially symmetric case for $n \geq 4$. The estimations of life span can be referred to [14, 15, 16, 17]

An efficient way to prove blowup results, when the critical exponent comes from the scaling properties of the partial differential operator, is the testing function method, first of all the test function method was introduce by Mitidieri-Pohozaev (see for example [18, 19, 20, 21]) and then applied by Zhang to study the critical case for the classical semilinear damped wave equation. In [22], they had used the smooth cutoff functions as the testing functions, and it seems enough to obtain a blowup result for Fujita type power. But if we want to get a blowup result for Strauss type power, it is better to use some special solutions of the linear wave equation as the testing function, i.e.,

$$\psi(t,x) = e^{-t} \int_{S^{n-1}} e^{x\cdot\omega} d\omega, \ n \geq 2, \tag{1.8}$$

used in [7].

Our aim is to study the exponent for the blowup to Eq.(1.1) with $\mu > 0, \ \alpha \in [0, 2)$ that is for given $n \geq 1$ and $p > 1$, the solutions of (1.1) will blow up in finite time when $1 < p \leq p_c(\mu, \alpha, n)$. If $\alpha > 2$, we guess it will have a global solution for any $p > 1$, we will give its proof in future papers.

The rest of the paper is organized as follows: in Section 2, we will state our main blowup results: Theorem 2.1-2.3. In Section 3, for $\mu \neq 1$, we use a key transformation to transform Eq.(1.1) into a Generalized Tricomi equation, which is introduced by D'Abbicco [11]. In Section 4, we define $F(t) = \int_{\mathbb{R}^n} u(t,x)dx$ as in [7] and use some modified Bessel functions (see [8]), and we choose a good testing function. We derive a Riccati-type ordinary differential inequality for $F(t)$ by a delicate analysis of Eq.(1.1). Especially in the critical case, we can use the fundamental solutions of the Generalized Tricomi equation (see [23]) to modify the Riccati-type ordinary differential inequality, we get a blowup result for Strauss type power. Almost repeating the proof in Section 4 can be similar to Theorem 2.2 in Section 5. If $\mu = 1$, Applying the testing function(see [22]) can be used to get a blowup result for Fujita type power, we shall complete the proof of Theorem 2.3 in Section 6.

2 Main results

In this paper, we say $f \lesssim g$ ($f \gtrsim g$), that means there exists a constant $C > 0$ such that $f \leq Cg$ ($f \geq Cg$). As in the introduction we denote throughout the article by $p_F(n)$ Fujita exponent

$$p_F(n) = 1 + \frac{2}{n}, \quad n \geq 1, \tag{2.1}$$

and $p_S(n)$ is called the Strauss index and is the positive root of the quadratic equation

$$(n-1)p^2 - (n+1)p - 2 = 0, \quad n \geq 2, \tag{2.2}$$

if $n = 1$, then $p_S(1) = \infty$. Similarly, we set

$$p_F(\mu, \alpha, n) = 1 + \frac{2-\alpha}{n+\mu-1}, \tag{2.3}$$

and $p_S(\mu, \alpha, n)$ is the positive root of the quadratic equation

$$(n-1+\mu)p^2 - (n+1+\mu-2\alpha)p - 2 = 0, \quad n \geq 1. \tag{2.4}$$

Let us state the main theorems that will be proved in the present article.

Theorem 2.1. *($0 \le \mu < 1$) For Eq.(1.1), if*

$$\begin{cases} (v_0(x), v_1(x)) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \text{ have compact supports}, \\ (v_0(x), v_1(x)) \text{ are non-negative and positive somewhere}, \end{cases} \tag{2.5}$$

and

$$p_{c_1}(\mu, \alpha, n) = max\{p_F(\mu, \alpha, n), p_S(\mu, \alpha, n)\}, \tag{2.6}$$

where $\mu \in [0, 1)$ and $\alpha \in [0, 2)$. Then the global solution $u \in C\left([0, \infty), H^1(\mathbb{R}^n)\right) \cap C^1\left([0, \infty), L^2(\mathbb{R}^n)\right)$ to Eq.(1.1) dose not exist provided that $1 < p \le p_{c_1}(\mu, \alpha, n)$.

Theorem 2.2. *($1 < \mu < 2$) For Eq.(1.1), if*

$$\begin{cases} (v_0(x), v_1(x)) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \text{ have compact supports}, \\ (v_0(x), v_1(x)) \text{ are non-negative and positive somewhere}, \end{cases} \tag{2.7}$$

and

$$p_{c_2}(\mu, \alpha, n) = max\{p_F(1, \alpha, n), p_S(\mu, \alpha, n)\}, \tag{2.8}$$

where $\mu \in (1, 2)$ and $\alpha \in [0, 2)$. Then the global solution $u \in C\left([0, \infty), H^1(\mathbb{R}^n)\right) \cap C^1\left([0, \infty), L^2(\mathbb{R}^n)\right)$ to Eq.(1.1) dose not exist provided that $1 < p \le p_{c_2}(\mu, \alpha, n)$.

Theorem 2.3. *($\mu = 1$) For Eq.(1.1), if*

$$\begin{cases} (v_0(x), v_1(x)) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \text{ have compact supports}, \\ \displaystyle\int_{\mathbb{R}^n} v_1(x)dx > 0, \end{cases} \tag{2.9}$$

where $\mu = 1$ and $\alpha \in [0, 2)$. Then the global solution $u \in C\left([0, \infty), H^1(\mathbb{R}^n)\right) \cap C^1\left([0, \infty), L^2(\mathbb{R}^n)\right)$ to Eq.(1.1) dose not exist provided that $1 < p \le p_F(1, \alpha, n)$.

Remark 2.1. If $\alpha = \mu = 0$, Eq.(1.1) returns to the classical wave equation, then $p_{c_1}(0, 0, n) = p_S(n)$, see [4, 5, 6, 7]. If $\alpha = 0$, Eq.(1.1) returns to the semilinear wave equation with scale invariant damping, then $p_{c_i}(0, 0, n) = p_\mu(n)$, i=1,2, see [15, 23].

Remark 2.2. If $\alpha > 2$, we guess it will have a global solution for any $p > 1$, we will give its proof in future articles.

3 Preliminaries

From [11], we will introduce some useful transformations. If $\mu \in (0, 1)$ in Eq.(1.1), by

$$u(t, x) = u(a(t) - 1, x), \tag{3.1}$$

where $a(t) = \frac{(1+t)^{k+1}}{k+1}$ and $k = \frac{\mu}{1-\mu}$, Cauchy problem (1.1) becomes a Cauchy problem for the Tricomi equation

$$\begin{cases} u_{tt} - (1+t)^{2k}\Delta u = c_k(1+t)^{2k-\alpha(k+1)}|u|^p, & \text{in} \quad (0, \infty) \times \mathbb{R}^n, \\ u(t_*, x) = v_0(x), & x \in \mathbb{R}^n, \\ u_t(t_*, x) = (1-\mu)^{-\mu} v_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{3.2}$$

where $t_* = (1-\mu)^{-(1-\mu)} - 1$ and $c_k = (k+1)^{-\alpha}$.

If $\mu \in (1,2)$ in Eq.(1.1), by

$$u(t,x) = a^{\mu-1}(t)u(a(t)-1,x), \tag{3.3}$$

where $a(t) = \frac{(1+t)^{k+1}}{k+1}$ and $k = \frac{2-\mu}{\mu-1}$, Cauchy problem (1.1) becomes a Cauchy problem for the Tricomi equation

$$\begin{cases} u_{tt} - (1+t)^{2k}\Delta u = c_k(1+t)^{2k-(p-1)-\alpha(k+1)}|u|^p, & \text{in} \quad (0,\infty)\times\mathbb{R}^n, \\ u(t_*,x) = v_0(x), & x \in \mathbb{R}^n, \\ u_t(t_*,x) = (1-\mu)^{2-\mu}\left(v_1(x) + (\mu-1)v_0(x)\right), & x \in \mathbb{R}^n, \end{cases} \tag{3.4}$$

where $t_* = (\mu-1)^{-(\mu-1)} - 1$ and $c_k = (\mu-1)^{(\mu-1)(p-1)}(k+1)^{-\alpha}$.

By the finite speed of propagation for Eq.(3.2) and

$$\text{supp}\{u_0(x), u_1(x)\} \subset \{x : |x| \le R\}. \tag{3.5}$$

Then

$$\text{supp}\{u(t,x)\} \subset \{x : |x| \le R + \phi(t) - \phi(0)\}, \tag{3.6}$$

where $\phi(t) = \frac{(1+t)^{k+1}}{k+1}$ and R is a constant from (3.5).

Introduce the following two useful functions: Following [7], the first one is

$$\begin{cases} \phi_1(x) = e^x + e^{-x}, \quad n = 1, \\ \phi_1(x) = \displaystyle\int_{\mathbb{R}^n} e^{x\cdot\omega} d\omega, \ n \ge 2, \end{cases} \tag{3.7}$$

which satisfies

$$\triangle\phi_1(x) = \phi_1(x). \tag{3.8}$$

From [7], we recall the following properties

Lemma 3.1. *([7]) If $\phi_1(x) = \int_{\mathbb{S}^{n-1}} e^{x\cdot\omega} d\omega$. Then*

$$\phi_1(x) \sim C(n)e^{|x|}|x|^{-\frac{n-1}{2}}, \quad as \quad |x| \to \infty. \tag{3.9}$$

The second one is the so-called modified Bessel function

$$I_\nu(t) = \int_0^\infty e^{-t\cosh z}\cosh(\nu z)dz, \ \nu \in \mathbb{R}, \tag{3.10}$$

and $I_\nu(t)$ is a solution of the equation

$$t^2\frac{d^2 I_\nu(t)}{dt^2} + t\frac{dI_\nu(t)}{dt} - (t^2+\nu^2)I_\nu(t) = 0, \ t > 0, \tag{3.11}$$

where ν is a real parameter. From [24], it follows that

(1) The asymptotic behavior of $I_\nu(t)$

$$I_\nu(t) = \sqrt{\frac{\pi}{2t}}e^{-t}\left(1 + O(t^{-1})\right) \text{ as } t \to \ \infty. \tag{3.12}$$

(2) The derivative identity

$$\frac{dI_\nu(t)}{dt} = -I_{\nu+1}(t) + \frac{\nu}{t}I_\nu(t). \tag{3.13}$$

Set

$$\lambda(t) = C(k)t^{\frac{1}{2}} I_{\frac{1}{2k+2}}\left(\frac{1}{k+1}t^{k+1}\right),\ t > 0, \tag{3.14}$$

where $C(k)$ is chosen so that $\lambda(t)$ satisfies

$$\begin{cases} \lambda''(t) - (1+t)^{2k}\lambda(t) = 0, & t \geq 0, \\ \lambda(0) = 1, \quad \lambda(\infty) = 0. \end{cases} \tag{3.15}$$

From [25]. Here is a list of properties of $\lambda(t)$.

Lemma 3.2. *([25]) From (3.12)-(3.14), it follows that*
(1)$\lambda(t)$ and $-\lambda'(t)$ are both decreasing, and

$$\lim_{t\to\infty} \lambda(t) = \lim_{t\to\infty} \lambda'(t) = 0 \tag{3.16}$$

(2) There exists a constant t_0 such that

$$\frac{1}{C_0}\lambda(t)t^k \leq |\lambda'(t)| \leq C_0\lambda(t)t^k, \quad \forall\ t \geq t_0, \tag{3.17}$$

where $C_0 = C_0(k, t_0)$.

Using Hölder's inequality, we have

$$\int_{\mathbb{R}^n} |u(t,x)|^p dx \geq |F_1(t)|^p \left(\int_{\mathbb{R}^n} \psi_1^{\frac{p}{p-1}}(t,x)\right)^{-(p-1)}, \tag{3.18}$$

where

$$F_1(t) = \int_{\mathbb{R}^n} \psi_1(t,x)u(t,x)dx, \tag{3.19}$$

$$\psi_1(t,x) = \lambda(t)\phi_1(x). \tag{3.20}$$

From [26], it is easy to get the following lemmas

Lemma 3.3. *([26]) Under the assumptions of Theorem 2.1, there exists a $t_0 > 0$ such that*

$$F_1(t) \geq C_1 t^{-k}, \quad \forall\ t \geq t_0, \tag{3.21}$$

where $C_1 = C_1(u_0, u_1, k, R, t_0)$.

Lemma 3.4. *([26]) By some properties of $\lambda(t)$ and $\phi_1(x)$, we deduce*

$$\begin{aligned} &\left(\int_{|x|\leq R+\phi(t)} \psi_1^{\frac{p}{p-1}}(t,x)dx\right)^{p-1} \\ &\leq C_2(1+t)^{(k+1)(n-1)(p-1)-\frac{1}{2}(k+1)(n-1)p-\frac{1}{2}kp}, \quad \forall\ t \geq t_0, \end{aligned} \tag{3.22}$$

where $C_2 = C_2(u_0, u_1, k, n, p, t_0, R)$.

It follows from (3.18), (3.21) and (3.22) that

$$\int_{\mathbb{R}^n} |u(t,x)|^p dx \geq C_3(1+t)^{\frac{p}{2}+(k+1)(n-1-\frac{np}{2})}, \quad \forall\ t \geq t_0, \tag{3.23}$$

where $C_3 = C_3(u_0, u_1, n, p, k_0, t_0, R)$. From [6, 7], we have the following lemma:

Lemma 3.5. ***(Kato's lemma)*** *Let $p > 1, q \in \mathbb{R}$ and $F \in C^2([0,T))$ be a positive function satisfying the nonlinear ordinary differential inequality*

$$\frac{d^2F(t)}{dt^2} \geq k_1(t+R)^{-q}F^p(t) \tag{3.24}$$

for any $t \in [T_1, T)$, for some $k_1, R > 0$ and $T_1 \in [0,T)$.
(1)If it holds the inequality

$$F(t) \geq k_0(t+R)^a \quad \text{for any} \quad t \in [T_0, T), \tag{3.25}$$

for some $a \geq 1$ satisfying $a > \frac{q-2}{p-1}$ and for some $k_0 > 0$ and $T_0 \in [0,T)$, then $T < \infty$.
(2) Let $q \geq p+1$ in (3.24) and suppose that the constant $k_0 = k_0(k_1) > 0$ is sufficiently large. Then, if (3.25) holds with $a = \frac{q-2}{p-1}$ for some $T_0 \in [0,T)$, then $T < \infty$.

4 The Proof of Theorem 2.1

Let us prove the blowup result for (3.2). Applying Lemma3.5 for the case in which the exponent a in (3.25) satisfies $a \geq \frac{q-2}{p-1}$, this condition corresponds to the requirement (2.6) in the statement of Theorem 2.1.

Proof of Theorem 2.1. **(1) The subcritical case**

In order to write simply, we put the initial time 0 instead of t_*. Recall (3.2) as

$$\begin{cases} u_{tt} - (1+t)^{2k}\Delta u = c_k(1+t)^{2k-\alpha(k+1)}|u|^p, & \text{in} \quad (0,\infty)\times\mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \\ u_t(0,x) = u_1(x), & x \in \mathbb{R}^n. \end{cases} \tag{4.1}$$

Set

$$F(t) = \int_{\mathbb{R}^n} u(t,x)dx. \tag{4.2}$$

Using (3.6), (4.1) and (4.2) and by integration by parts, we get

$$\ddot{F}(t) = c_k(1+t)^{2k-\alpha(k+1)}\int_{\mathbb{R}^n}|u(t,x)|^p dx, \tag{4.3}$$

then (3.23) gives

$$\ddot{F}(t) \geq C_4(1+t)^{\frac{p}{2}+(k+1)(n-1-\frac{np}{2})+2k-\alpha(k+1)}, \quad \forall\, t \geq t_0, \tag{4.4}$$

where $C_4 = C_4(u_0, u_1, t_0, k, \alpha, n, p, R)$. Integrating twice the previous relation leads to

$$F(t) \geq C_5(1+t)^{\max\{\frac{p}{2}+(k+1)(n-1-\frac{np}{2})+2k-\alpha(k+1)+2,1\}}, \quad \forall\, t \geq t_0, \tag{4.5}$$

where $C_5 = C_5(u_0, u_1, t_0, k, \alpha, n, p, R)$. In view of (4.3), (3.6) and Hölder's inequality, we get

$$\ddot{F}(t) \geq C_6(1+t)^{-(k+1)n(p-1)+2k-\alpha(k+1)}F^p(t), \tag{4.6}$$

where $C_6 = C_6(k, n, t_0, p, R)$.

If $a = \frac{p}{2} + (k+1)(n-1-\frac{np}{2}) + 2k - \alpha(k+1) + 2$ and $q = (k+1)n(p-1) - 2k + \alpha(k+1)$, then applying Lemma3.5, we have

$$\frac{p}{2} + (k+1)(n-1-\frac{np}{2}) + 2k - \alpha(k+1) + 2 > (k+1)n + \frac{\alpha(k+1)-2k-2}{p-1},$$

using $k = \frac{\mu}{1-\mu}$, then

$$(n + \mu - 1)p^2 - (n + \mu + 1 - 2\alpha)p - 2 < 0. \tag{4.7}$$

Precisely

$$1 < p < p_S(\mu, \alpha, n). \tag{4.8}$$

If $a = 1$ and $q = (k + 1)n(p - 1) - 2k + \alpha(k + 1)$, then applying Lemma 3.5, we have

$$1 > (k + 1)n + \frac{\alpha(k + 1) - 2k - 2}{p - 1}, \tag{4.9}$$

so

$$1 < p < p_F(\mu, \alpha, n). \tag{4.10}$$

From (4.8) and (4.10), we get

$$1 < p < p_{c_1}(\mu, \alpha, n), \tag{4.11}$$

the solutions to (1.1) will blow up in finite time.

(2) The critical case

If $p = p_F(\mu, \alpha, n)$, it's easy to have

$$-(k + 1)n(p - 1) + 2k - \alpha(k + 1) + p = -1. \tag{4.12}$$

(4.6) and (4.12) give

$$\begin{aligned} \ddot{F}(t) &\geq C_6(1 + t)^{-(k+1)n(p-1)+2k-\alpha(k+1)} F^p(t) \\ &\geq C_7(1 + t)^{-(k+1)n(p-1)+2k-\alpha(k+1)+p} \\ &= C_7(1 + t)^{-1}, \quad \forall\ t \geq t_0, \end{aligned} \tag{4.13}$$

where $C_7 = C_7(k, n, t_0, p, u_0, u_1, R)$. Intergrading twice the previous relation leads to

$$F(t) \geq C_8(t + 1)\ln(1 + t), \quad \forall\ t \geq t_0, \tag{4.14}$$

where $C_8 = C_8(u_0, u_1, k, n, t_0, p, R)$. So

$$F(t) \geq k_0(1 + t), \tag{4.15}$$

for large $t > 0$ and k_0 is sufficiently large, then applying Lemma3.5, we know $p = p_F(\mu, \alpha, n)$ also in the range of blowup.

If $p = p_S(\mu, \alpha, n)$, then

$$(n(k + 1) - 1)\, p^2 - ((k + 1)(n + 2 - 2\alpha) - 1)\, p - 2(k + 1) = 0. \tag{4.16}$$

Step1: With no loss of generality we assume that $u(t, \cdot)$ is radial. This because

$$\bar{u}_{tt} - (1 + t)^{2k} \triangle \bar{u} \geq c_k (1 + t)^{2k-\alpha(k+1)} |\bar{u}|^p, \tag{4.17}$$

where

$$\bar{u} = \frac{1}{\omega_n} \int_{S^{n-1}} u(t, r, \theta) d\theta$$

is the spherical average of u.

Step2: The lower bound of $R(u)$. The practices of reference [27, 23], let $\omega \in \mathbb{R}^n$ be a unit vector. The Radon transform of u with respect to the space variables is defined a

$$R(u)(t,\rho) = \int_{\{x:\ x\cdot\omega=\rho\}} u(t,x)dS_x, \tag{4.18}$$

where dS_x is the Lebesque measure on the hyper-plane $\{x : x \cdot w = 0\}$. Next we show that $R(u)$ is a function of ρ and t and is in fact independent of ω, when u is radially symmetric. From (4.18), it's easy to see

$$\begin{aligned} R(u)(t,\rho) =& \int_{\{x':\ x'\cdot\omega=0\}} u(t,\rho\omega + x')dS_{x'} \\ =& c_n \int_0^\infty u(t, \sqrt{\rho^2 + x'^2})|x'|^{n-2} d|x'| \\ =& c_n \int_{|\rho|}^\infty u(r,t)(r^2-\rho^2)^{\frac{n-3}{2}} r dr, \end{aligned} \tag{4.19}$$

where c_n is a constant. This shows that $R(u)(t,\rho)$ is independent of ω. From [28], we deduce

$$R(\triangle u)(t,\rho) = \partial_\rho^2 R(u)(t,\rho). \tag{4.20}$$

Since u is a solution to (4.1), it's well known that $R(u)$ satisfies one-dimensional Generalized Tricomi equation

$$\begin{cases} \partial_t^2 R(u)(t,\rho) - (1+t)^{2k}\partial_\rho^2 R(u)(t,\rho) = c_k(1+t)^{2k-\alpha(k+1)} R(|u|^p)(t,\rho), \\ R(u)(0,x) = R(u_0(\rho)), \\ \partial_t R(u)(0,x) = R(u_1(\rho)). \end{cases} \tag{4.21}$$

Set

$$\phi(t) = \frac{(1+t)^{k+1}}{k+1}, \quad A(t) = \phi(t) - \phi(0), \tag{4.22}$$

then

$$\text{supp } u(t,\cdot) \subset [-(R + A(t)), R + A(t)]. \tag{4.23}$$

From [23], we have

$$\begin{aligned} R(u)(t,\rho) =& \frac{1}{2}(1+t)^{-\frac{k}{2}}(f(\rho + A(t)) + f(\rho - A(t))) \\ &+ \int_0^{A(t)} (f(\rho-\sigma) + f(\rho+\sigma))K_0(t,\sigma)d\sigma \\ &+ \int_0^{A(t)} (g(\rho-\sigma) + g(\rho+\sigma))K_1(t,\sigma)d\sigma \\ &+ C\int_0^t \int_0^{A(t)-A(s)} (1+s)^{2k-\alpha(k+1)} \left[|u(b,\rho-\sigma)|^p + |u(b,\rho+\sigma)|^p\right] \\ &\qquad \times E(t,\sigma;s,0)d\sigma ds, \end{aligned} \tag{4.24}$$

where

$$\begin{aligned} &E(t,\sigma;b,0) = ((\phi(t)+\phi(b))^2 - \sigma^2)^{-\gamma} \times F(\gamma,\gamma,1,z), \\ &K(t,\sigma) = CE(t,\sigma;0,0), \quad K_0(t,\sigma) = -C\frac{\partial E(t,\sigma;s,0)}{\partial s}\bigg|_{s=0}, \\ &z = \frac{(\phi(t)-\phi(s))^2 - (\rho-\sigma)^2}{(\phi(t)+\phi(s))^2 - (\rho-\sigma)^2} \in [0,1),\ \gamma = \frac{k}{2(k+1)}, \end{aligned}$$

and $F(\gamma,\gamma,1,z)$ is the hypergeometric function (see [24]). It's easy to see

$$\begin{aligned} F(\gamma,\gamma,1,z) &= \frac{1}{\Gamma(\gamma)\Gamma(1-\gamma)}\int_0^1 s^{\gamma-1}(1-s)^{-\gamma}(1-zs)^{-\gamma}ds \\ &\geq \frac{1}{\Gamma(\gamma)\Gamma(1-\gamma)}B(\gamma,1-\gamma)=1 \end{aligned} \tag{4.25}$$

From the assumptions that the initial data of u are nonnegative, we can get

$$\begin{aligned} R(u)(t,\rho) \geq & c_k\int_0^t\int_{\rho-(A(t)-A(s))}^{\rho+(A(t)-A(s))}\left((A(t)+A(s))^2-(\rho-\sigma)^2\right)^{-\gamma} \\ & \times(1+s)^{2k-\alpha(k+1)}R(|u|^p)(s,\sigma)d\sigma ds. \end{aligned} \tag{4.26}$$

Note that the support of $R(u)(\cdot,s)$ is contained in $B(0,\phi(s)+R)$. From now on we will assume $\rho\geq 0$, unless stated otherwise. If $A(s)\leq A(s_1)=\frac{A(t)-\rho-R}{2}$, then

$$\rho+(A(t)-A(s))\geq A(s)+R,\quad \rho-(A(t)-A(s))\leq -(A(s)+R). \tag{4.27}$$

From (4.26) and (4.27) it follows that

$$\begin{aligned} R(u)(t,\rho) &\geq c_k\int_0^{s_1}\int_{R+A(s)}^{-(R+A(s))}\left((\phi(t)+\phi(s))^2-(\rho-\sigma)^2\right)^{-\gamma} \\ &\qquad\times(1+s)^{2k-\alpha(k+1)}R(|u|^p)(s,\sigma)d\sigma ds \\ &= c_k\int_0^{s_1}\int_{-\infty}^{+\infty}\left((\phi(t)+\phi(s))^2-(\rho-\sigma)^2\right)^{-\gamma} \\ &\qquad\times(1+s)^{2k-\alpha(k+1)}R(|u|^p)(s,\sigma)d\sigma ds. \end{aligned} \tag{4.28}$$

By (4.27) we have

$$\phi(t)+\phi(s)+\rho-\sigma\leq 2\phi(t),\quad \phi(t)+\phi(s)-(\rho-\sigma)\leq 2(\phi(t)-\rho). \tag{4.29}$$

Using (4.28) and (4.29) gives

$$\begin{aligned} R(u)(t,\rho) &\geq C_9\int_0^{s_1}\int_{-\infty}^{+\infty}(\phi(t)-\rho)^{-\gamma}\phi^{-\gamma}(t)(1+s)^{2k-\alpha(k+1)}R(|u|^p)(s,\sigma)d\sigma ds \\ &= C_9(\phi(t)-\rho)^{-\gamma}\phi^{-\gamma}(t)\int_0^{s_1}\int_{-\infty}^{+\infty}(1+s)^{2k-\alpha(k+1)} \\ &\qquad\times R(|u|^p)(s,\sigma)d\sigma ds \\ &= C_9(\phi(t)-\rho)^{-\gamma}\phi^{-\gamma}(t)\int_0^{s_1}\int_{\mathbb{R}^n}(1+s)^{2k-\alpha(k+1)} \\ &\qquad\times|u|^p(s,x)dxds \\ &= C_9(\phi(t)-\rho)^{-\gamma}\phi^{-\gamma}(t)\int_0^{s_1}\ddot{F}(s)ds, \end{aligned} \tag{4.30}$$

where $C_9=2^{-2\gamma}c_k$. Since $\frac{p}{2}+(k+1)(n-1-\frac{np}{2})+2k-\alpha(k+1)>-1$ and applying (4.4), we can get

$$\begin{aligned} R(u)(t,\rho) &\gtrsim (\phi(t)-\rho)^{-\gamma}\phi^{-\gamma}(t)\int_0^{s_1}(1+s)^{\frac{p}{2}+(k+1)(n-1-\frac{np}{2})+2k-\alpha(k+1)}ds \\ &\gtrsim (\phi(t)-\rho)^{-\gamma}\phi^{-\gamma}(t)(1+s_1)^{\frac{p}{2}+(k+1)(n-1-\frac{np}{2})+2k-\alpha(k+1)+1} \\ &\gtrsim (\phi(t)-\rho)^{-\gamma}\phi^{-\gamma}(t)(A(t)-\rho-R)^{\frac{p-2}{2(k+1)}+2-\alpha+n-1-\frac{np}{2}}. \end{aligned} \tag{4.31}$$

Step 3 The lower bound of $\int_{\mathbb{R}^n} |u(t,x)|^p dx$. From [7], one can introduce the transformation

$$T(f)(\rho) = \frac{1}{|A(t)-\rho+R|^{\frac{n-1}{2}}} \int_{\rho}^{A(t)+R} f(r)|r-\rho|^{\frac{n-3}{2}} dr \tag{4.32}$$

and further derive

$$||T(f)||_{L^p} \le C||f||_{L^p}, \tag{4.33}$$

where C is a constant. In fact, if $n \ge 3$, then

$$|T(f)| \le \frac{2}{2|A(t)-\rho+R|} \int_{2\rho-(A(t)+R)}^{A(t)+R} |f(r)|dr \le 2M(|f|)(\rho), \tag{4.34}$$

where$M(|f(x)|)$ is the maximal function of $f(x)$, so (4.33) holds.

For $n = 2$, at first we prove that T maps L^∞ to L^∞ and L^1 to $L^{1,w}$(weak L^1 space), by Marcinkiewicz interpolation theorem, then (4.33) holds for $n = 2$.

In fact, for $n = 2$, we have

$$\begin{aligned} |T(f)(\rho)| =& \frac{1}{|A(t)-\rho+R|^{\frac{1}{2}}} \int_0^{A(t)+R} f(r)|r-\rho|^{-\frac{1}{2}} dr \\ \le& \frac{||f||_{L^\infty([0,A(t)+R])}}{|A(t)-\rho+R|^{\frac{1}{2}}} \int_0^{A(t)+R} |r-\rho|^{-\frac{1}{2}} dr \\ =& 2||f||_{L^\infty([0,A(t)+R])} \frac{1}{|A(t)-\rho+R|^{\frac{1}{2}}} |A(t)-\rho+R|^{\frac{1}{2}} \\ =& 2||f||_{L^\infty([0,A(t)+R])}, \end{aligned} \tag{4.35}$$

which yields the $L^\infty - L^\infty$ estimate of operator T. Next we derive the $L^1 - L^{1,w}$ estimate of T. Suppose $f \in L^1([0, A(t)+R])$. Let

$$g(\rho) = \frac{1}{|A(t)-\rho+R|^{\frac{1}{2}}}, \quad h(\rho) = \int_{\rho}^{A(t)+R} f(r)|r-\rho|^{-\frac{1}{2}} dr, \tag{4.36}$$

Denote $d_\varphi = |\{0 \le \rho \le A(t)+R : \varphi(\rho) > \alpha\}|$ as the distribution function of φ. It's known that for $0 < \alpha < \infty$ and measurable functions f_1, f_2,

$$d_{f_1,f_2}(\alpha) \le d_{f_1}(\alpha^{\frac{1}{2}}) + d_{f_2}(\alpha^{\frac{1}{2}}). \tag{4.37}$$

Note that

$$d_g(\alpha^{\frac{1}{2}}) = |\{0 \le \rho \le A(t)+R : g(\rho) > \alpha\}| = \frac{1}{\alpha}. \tag{4.38}$$

In addition,

$$|h(\rho)| \le \int_0^{A(t)+R} |f(r)||r-\rho|^{-\frac{1}{2}} dr = f * \frac{1}{|r|^{\frac{1}{2}}}. \tag{4.39}$$

Since $\frac{1}{|r|^{\frac{1}{2}}} \in L^{2,w}([0, A(t)+R])$ and $f \in L^1([0, A(t)+R])$, by Young's inequality, we have $h \in L^{2,w}([0, A(t)+R])$. Therefore,

$$\alpha d_{gh}(\alpha) \le \alpha d_f(\alpha^{\frac{1}{2}}) + \alpha d_h(\alpha^{\frac{1}{2}}) \le C, \tag{4.40}$$

which means $T(f)(\rho) = g(\rho)h(\rho) \in L^{1,w}([0, A(t) + R])$. Then an application of Marcinkiewicz interpolation theorem yields

$$||T(f)||_{L^p([0,A(t)+R])} \leq C||f||_{L^p([0,A(t)+R])}, \tag{4.41}$$

where $C > 0$ is a uniform constant independent of t.

Using (4.33) to the function

$$f(r) = \begin{cases} |u(t,r)|r^{\frac{n-1}{p}}, & r \geq 0, \\ 0, & r < 0, \end{cases} \tag{4.42}$$

then

$$\begin{aligned} &\int_0^{A(t)+R-1} \left(\frac{1}{|A(t)-\rho+R|^{\frac{n-1}{2}}} \int_\rho^{A(t)+R} |u(t,r)|r^{\frac{n-1}{p}}|r-\rho|^{\frac{n-3}{2}} dr \right)^p d\rho \\ &\lesssim \int_0^\infty |u(t,r)|^p r^{n-1} dr = \int_{\mathbb{R}^n} |u(t,x)|^p dx. \end{aligned} \tag{4.43}$$

For $\rho \leq r \leq A(t) + R$, it holds

$$r^{\frac{n-1}{p}} \geq \begin{cases} r^{\frac{n-1}{2}} \rho^{\frac{n-1}{p} - \frac{n-1}{2}}, & 1 < p \leq 2, \\ r^{\frac{n-1}{2}} (A(t) - R - 1)^{\frac{n-1}{p} - \frac{n-1}{2}}, & p > 2. \end{cases} \tag{4.44}$$

Now, we only treat the case of $1 < p \leq 2$ since the treatment for $p > 2$ is completely similar. When $1 < p \leq 2$, from (4.43) and (4.44) it follows that

$$\begin{aligned} &\int_0^{A(t)+R-1} \left(\frac{1}{|A(t)-\rho+R|^{\frac{n-1}{2}}} \int_\rho^{A(t)+R} |u(t,r)|r^{\frac{n-1}{2}}|r-\rho|^{\frac{n-3}{2}} dr \right)^p \\ &\qquad \times \rho^{(n-1)(1-\frac{p}{2})} d\rho \lesssim \int_{\mathbb{R}^n} |u(t,x)|^p dx. \end{aligned} \tag{4.45}$$

Since supp $u(t, \cdot) \subset B(0, A(t) + R)$, it's easy to see

$$\begin{aligned} R(u)(t,\rho) &\lesssim \int_{|\rho|}^{A(t)+R} |u(r,t)|(r^2-\rho^2)^{\frac{n-3}{2}} r dr \\ &\lesssim \int_{|\rho|}^{A(t)+R} |u(r,t)|(r+\rho)^{\frac{n-3}{2}}(r-\rho)^{\frac{n-3}{2}} r dr \\ &\lesssim \int_{|\rho|}^{A(t)+R} |u(r,t)|r^{\frac{n-1}{2}}(r-\rho)^{\frac{n-3}{2}} r dr. \end{aligned} \tag{4.46}$$

Substituting (4.46) into (4.45) leads to

$$\int_0^{A(t)+R} \frac{(R(u)(t,\rho))^p}{|A(t)-\rho+R|^{\frac{(n-1)p}{2}}} \rho^{(n-1)(1-\frac{p}{2})} d\rho \lesssim \int_{\mathbb{R}^n} |u(t,x)|^p dx. \tag{4.47}$$

If $\rho \in (0, A(t) - R - 1)$, then $\phi(t) > 2(R + 1)$ and

$$A(t) - \rho + R \leq C(A(t) - \rho - R), \quad A(t) - \rho \leq C(A(t) - \rho - R), \tag{4.48}$$

where C is a constant. Using (4.31), (4.47) and (4.48), we find that

$$\begin{aligned} &\int_0^{A(t)-R-1} \frac{\rho^{(n-1)(1-\frac{p}{2})}\phi^{-\gamma p}(t)}{|A(t)-\rho+R|^{\beta p + \frac{(n-1)p}{2} - (\frac{p-2}{2(k+1)} + 2 - \alpha + n - 1 - \frac{np}{2})p}} d\rho \\ &\lesssim \int_{\mathbb{R}^n} |u(t,x)|^p dx. \end{aligned} \tag{4.49}$$

Since $p = p_S(\mu, \alpha, n)$ it holds $\gamma p + \frac{(n-1)p}{2} - \left(\frac{p-2}{2(k+1)} + 2 - \alpha + n - 1 - \frac{np}{2}\right) p = 1$, where $\gamma = \frac{k}{2(k+1)}$. Then (4.49) becomes

$$\begin{aligned} \int_{\mathbb{R}^n} |u(t,x)|^p dx &\gtrsim \int_0^{A(t)-R-1} \frac{\rho^{(n-1)(1-\frac{p}{2})}\phi^{-\gamma p}(t)}{A(t)-\rho-R} d\rho \\ &\gtrsim \phi^{(n-1)(1-\frac{p}{2})-\gamma p}(t) \int_{\frac{A(t)-R-1}{2}}^{A(t)-R-1} \frac{1}{A(t)-\rho-R} d\rho \\ &\gtrsim (1+t)^{(k+1)(n-1)(1-\frac{p}{2})-\frac{kp}{2}} \ln(1+t), \end{aligned} \tag{4.50}$$

for large $t > 0$. Thus

$$\begin{aligned} \ddot{F}(t) &= c_k (1+t)^{2k-\alpha(k+1)} \int_{\mathbb{R}^n} |u(t,x)|^p dx \\ &\gtrsim (1+t)^{(k+1)(n-1)(1-\frac{p}{2})-\frac{kp}{2}+2k-\alpha(k+1)} \ln(1+t), \end{aligned} \tag{4.51}$$

for large $t > 0$. It can be obtained by twice integrations on $[0, t]$

$$F(t) \gtrsim F(0) + \dot{F}(0)t + (1+t)^{(k+1)(n-1)(1-\frac{p}{2})-\frac{kp}{2}+2k+2-\alpha(k+1)} \ln(1+t), \tag{4.52}$$

for large $t > 0$. Thus

$$F(t) \leq k_0 (1+t)^{(k+1)(n-1)(1-\frac{p}{2})-\frac{kp}{2}+2k+2-\alpha(k+1)}, \tag{4.53}$$

for large $t > 0$ and k_0 is sufficiently large. Applying Lemma3.5, we know $p = p_S(\mu, \alpha, n)$ also in the range of blowup. $\square$

5 The Proof of Theorem 2.2

Let us prove the blowup result for (3.4), Using Lemma3.5 for the case in which the exponent a in (3.25) satisfies $a \geq \frac{q-2}{p-1}$, this condition corresponds to the requirement (2.8) in the statement of Theorem2.2.

Proof of Theorem 2.2. For simple writing, we put initial time 0 instead of t_*. Recall (3.4) as

$$\begin{cases} u_{tt} - (1+t)^{2k}\Delta u = c_k (1+t)^{2k-\alpha(k+1)-(p-1)} |u|^p, & \text{in} \quad (0,\infty)\times\mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \\ u_t(0,x) = u_1(x), & x \in \mathbb{R}^n. \end{cases} \tag{5.1}$$

By repeating the process of proving Theorem 2.1, we can prove Theorem 2.2. We can get $a = \frac{p}{2} + (k+1)(n-1-\frac{np}{2}) + 2k - \alpha(k+1) - (p-1) + 2$ and $q = (k+1)n(p-1) - 2k + \alpha(k+1) + (p-1)$, then applying Lemma 3.5, we know if $1 < p \leq p_{c_2}(\mu, \alpha, n)$, the solutions will blow up in finite time. $\square$

6 The Proof of Theorem 2.3

In this section, we will prove Theorem 2.3, our strategy is the testing function argument, which was builded by Zhang [22] can be employed, more More details, we can see [29, 30, 9].

Proof of Theorem 2.3. We introduce the test function depending on the parameter $R > 0$

$$\psi_R(t,x) = \eta_R(t)\phi_R(r) = \eta(\frac{t}{R})\phi(\frac{r}{R}), \quad \text{for} \quad |x| = r, \tag{6.1}$$

where $\eta(t),\ \phi(r) \in C_0^\infty$ satisfy

$$0 \le \eta(t) \le 1, \qquad |\eta'(t)|, |\eta''(t)| \le C, \qquad \frac{(\eta'(t))^2}{\eta(t)} \le C,$$

$$\eta(t) = \begin{cases} 1 & t \in [0, \frac{1}{2}], \\ 0 & t \in [1, \infty), \end{cases}$$

$$0 \le \phi(r) \le 1, \qquad |\phi'(r)|, |\phi''(r)| \le C, \qquad \frac{(\nabla\phi(r))^2}{\eta(t)} \le C,$$

$$\phi(r) = \begin{cases} 1 & t \in [0, \frac{1}{2}], \\ 0 & t \in [1, \infty). \end{cases}$$

Recall Eq.(1.1) as

$$v_{tt} - \Delta v + \frac{1}{1+t} v_t = (1+t)^{-\alpha}|v|^p, \ (\mu = 1). \tag{6.2}$$

Multiplying (6.2) by some C^2 function $g(t) > 0$, we derive

$$(gv)_{tt} - \Delta(gv) - (g'v)_t + (-g' + gb_1)v_t = (1+t)^{-\alpha} g|v|^p, \tag{6.3}$$

where $b_1(t) = \frac{1}{1+t}$. If $-g' + gb = 0$ for $t > 0$ and $g(0) > 0$, then

$$g(t) = g(0)(1+t). \tag{6.4}$$

So

$$(gv)_{tt} - \Delta(gv) - (g'v)_t = (1+t)^{-\alpha} g|v|^p. \tag{6.5}$$

Define

$$I_R = \int_{Q_R} (1+t)^{-\alpha} g(t)|v|^p \psi_R^q dxdt, \tag{6.6}$$

where $Q_R = [0, R] \times B_R,\ B_R = \{x \in \mathbb{R}^n : |x| \le R\}$ and $\frac{1}{p} + \frac{1}{q} = 1$. By integration by parts, we can get

$$\begin{aligned} I_R =& -g(0)\int_{B_R} v_1(x)\phi_R^q(r)dx \\ &+ \int_{Q_R} g(t)v(t,x)(\psi_R^q(t,x))_{tt} dxdt \\ &+ \int_{Q_R} \left(g'(t)v(t,x)\right)(\psi_R^q(t,x))_t dxdt \\ &- \int_{Q_R} g(t)v(t,x)\Delta(\psi_R^q(t,x))dxdt \\ =& -g(0)\int_{B_R} v_1(x)\phi_R^q(r)dx + J_1 + J_2 + J_3. \end{aligned} \tag{6.7}$$

where

$$J_1 = \int_{Q_R} g(t)v(t,x)(\psi_R^q(t,x))_{tt} dxdt,$$

$$J_2 = \int_{Q_R} (g'(t)v(t,x))(\psi_R^q(t,x))_t dxdt, J_3 = -\int_{Q_R} g(t)v(t,x)\Delta(\psi_R^q(t,x))dxdt.$$

In view of assumption (2.9), then

$$I_R < \sum_{i=1}^{3} J_i. \tag{6.8}$$

Using Hölder's inequality and the previous inequality about $\eta(t)$, $\phi(r)$ to estimate $J_i(i = 1, 2, 3)$, we get

$$\begin{aligned}|J_2| &= \left|\int_{Q_R} g(0)v(t,x)(\psi_R^q(t,x))_t\right| dxdt \\ &\lesssim R^{-1}\int_{\hat{Q}} |v(t,x)|\psi_R^{q-1}(t,x)dxdt \\ &\lesssim R^{-1}\hat{I}_R^{\frac{1}{p}}\left(\int_{\hat{Q}_R}(1+t)^{(\alpha-1)\frac{q}{p}}dxdt\right)^{\frac{1}{q}} \\ &\lesssim R^{\frac{-q(2-\alpha)+n+2-\alpha}{q}}\hat{I}_R^{\frac{1}{p}}.\end{aligned} \tag{6.9}$$

Similarly

$$|J_1| \lesssim R^{-2}\int_{\hat{Q}_R} g(t)|v(t,x)|\psi_R^{q-1}(t,x)dxdt \lesssim R^{\frac{-q(2-\alpha)+n+2-\alpha}{q}}\hat{I}_R^{\frac{1}{p}}, \tag{6.10}$$

$$|J_3| \lesssim R^{-2}\int_{\tilde{Q}_R} g(t)|v(t,x)|\psi_R^{q-1}(t,x)dxdt \lesssim R^{\frac{-q(2-\alpha)+n+2-\alpha}{q}}\tilde{I}_R^{\frac{1}{p}}, \tag{6.11}$$

where

$$\hat{I}_R = \int_{\hat{Q}_R}(1+t)^{-\alpha}g(t)|v(t,x)|^p\psi_R^q(t,x)dxdt,$$

$$\tilde{I}_R = \int_{\tilde{Q}_R}(1+t)^{-\alpha}g(t)|v(t,x)|^p\psi_R^q(t,x)dxdt,$$

and $\hat{Q}_R = \left[\frac{R}{2}, R\right] \times B_R(0), \tilde{Q}_R = [0,R] \times (B_{\frac{R}{2}}(0), B_R(0))$. It follows from (6.8)-(6.11) that

$$I_R \lesssim (\tilde{I}_R^{\frac{1}{p}} + \hat{I}_R^{\frac{1}{p}} + I_R^{\frac{1}{p}})R^{\frac{-q(2-\alpha)+n+2-\alpha}{q}} \lesssim I_R^{\frac{1}{p}}R^{\frac{-q(2-\alpha)+n+2-\alpha}{q}}, \tag{6.12}$$

which impels

$$I_R^{1-\frac{1}{p}} \lesssim R^{\frac{-q(2-\alpha)+n+2-\alpha}{q}}. \tag{6.13}$$

If $0 < p < p_F(1, \alpha, , n)$, we have $I_R \to 0$ as $R \to \infty$, then $v \equiv 0$, therefore, we have $\int_{\mathbb{R}^n} v_1(x)dx = 0$, which contradicts the assumption on the data of (2.9).

If $p = p_F(1, \alpha, n)$, we have $I_R \leq C$, with some constant C independent of R, so

$$\lim_{R\to\infty}(\tilde{I}_R + \hat{I}_R) = 0, \quad \text{then} \quad \lim_{R\to\infty} I_R = 0.$$

Therefore $v \equiv 0$, it also leads a contradiction. □

7 Conclusions

In this study We obtain a blowup result for solutions to a semilinear wave equation with scale-invariant dissipation. We perform a change of variables that transforms our starting equation into a Generalized Tricomi equation, then apply Kato's lemma, we can prove a blowup result for solutions to the transformed equation under some assumptions on the initial data.

Competing Interests

Author has declared that no competing interests exist.

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